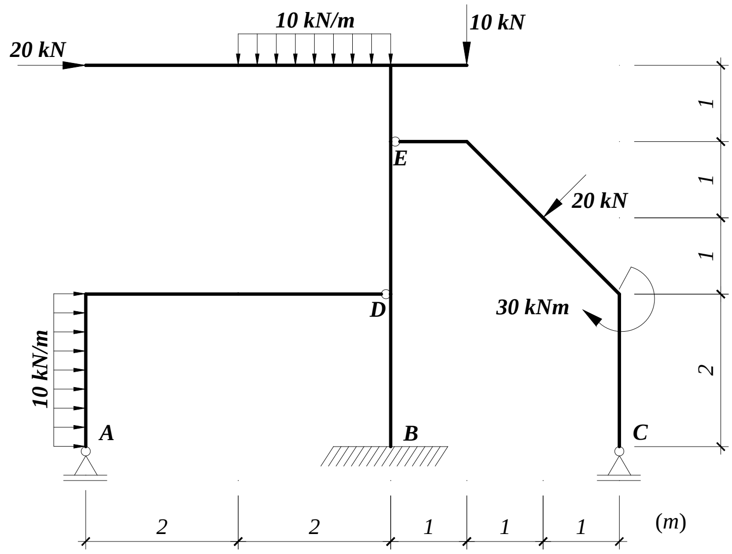
VJEŽBA BR. 1

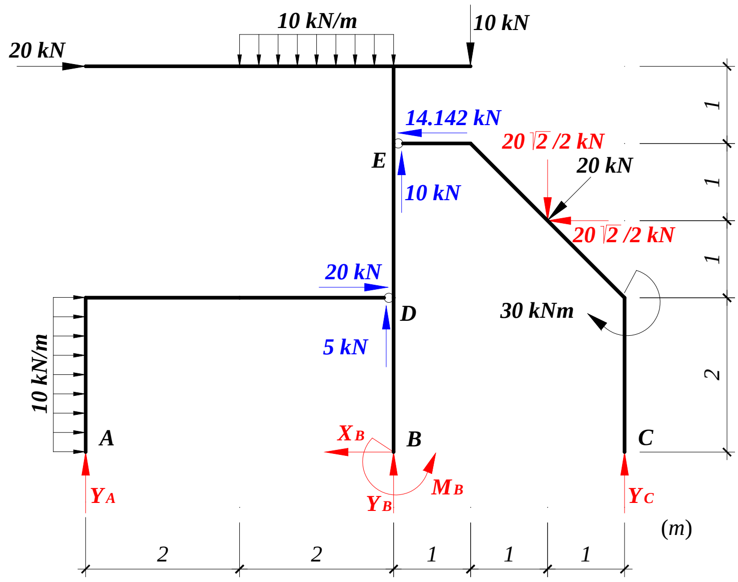
PONAVLJANJE GRADIVA IZ GRAĐEVINSKE MEHANIKE 1

RAVANSKI NOSAČI

1. Za nosač na slici nacrtati dijagrame presječnih sila (*M*, *T*, *N*).



R J E Š E NJ E

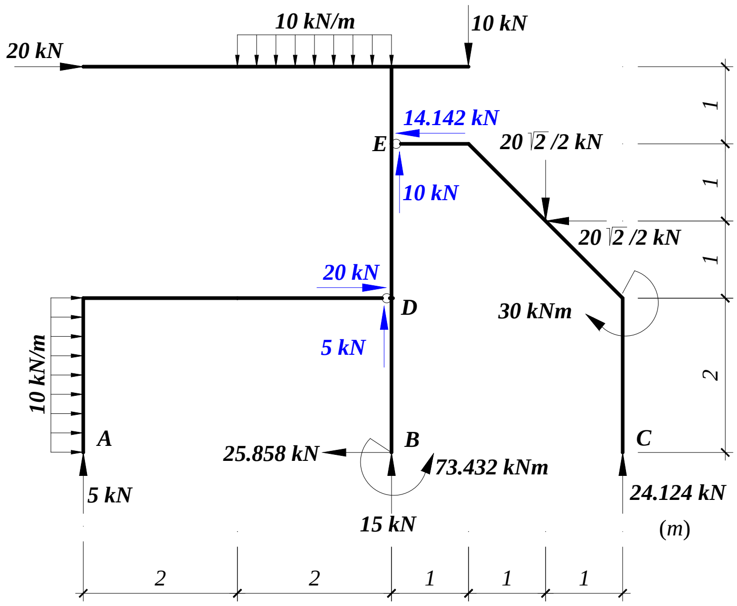


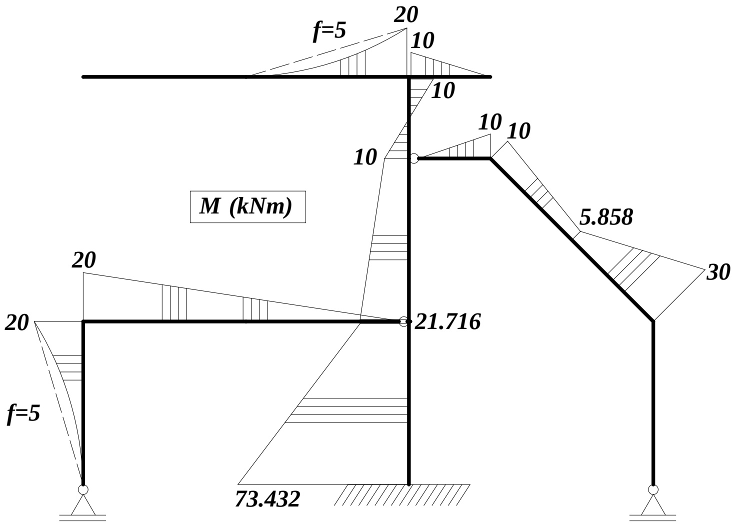
Uslovi ravnoteže:

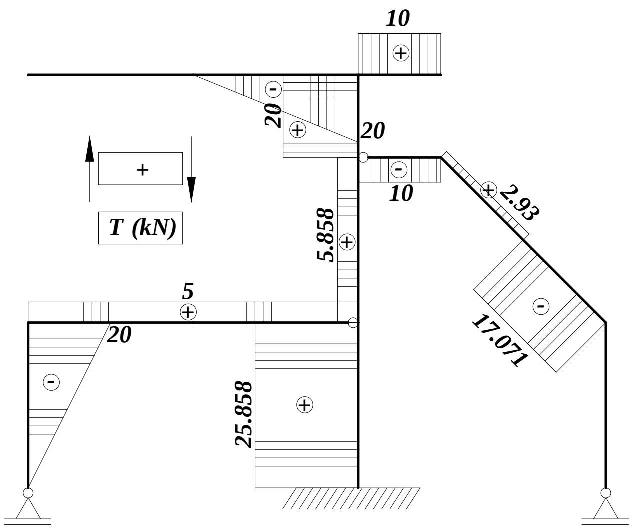
1. $\sum X = 0 \quad \rightarrow X_B = 20 - \frac{20\sqrt{2}}{2} + 10 \cdot 2 \quad \rightarrow X_B = 25.858\ kN$
2. $\sum M_{D,\ lijevo} = 0 \quad \rightarrow Y_A \cdot 4 - 10 \cdot 2 \cdot 1 = 0 \quad \rightarrow Y_A = 5.0\ kN$
3. $\sum M_{E,\ desno} = 0 \quad \rightarrow Y_C \cdot 3 - 30 - \frac{20\sqrt{2}}{2} \cdot (2 + 1) = 0 \quad \rightarrow Y_C = 24.142\ kN$
4. $\sum Y = 0 \quad \rightarrow Y_A + Y_B + Y_C - 10 \cdot 2 - 10 - \frac{20\sqrt{2}}{2} = 0 \quad \rightarrow Y_B = 15.0\ kN$

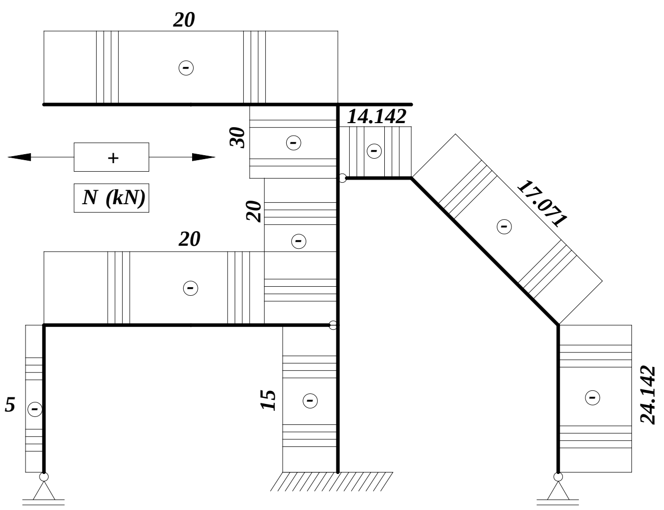
5. $\sum M_B = 0$ $\rightarrow$ $20 \cdot 5 + 10 \cdot 1 - 10 \cdot 2 \cdot 1 + 20 \cdot 2 - 14.142 \cdot 4 = M_B$ $\rightarrow$ $M_B = 73.432\ kN$

Kontrola: npr. $\sum M_A = 0$ – zadovoljeno

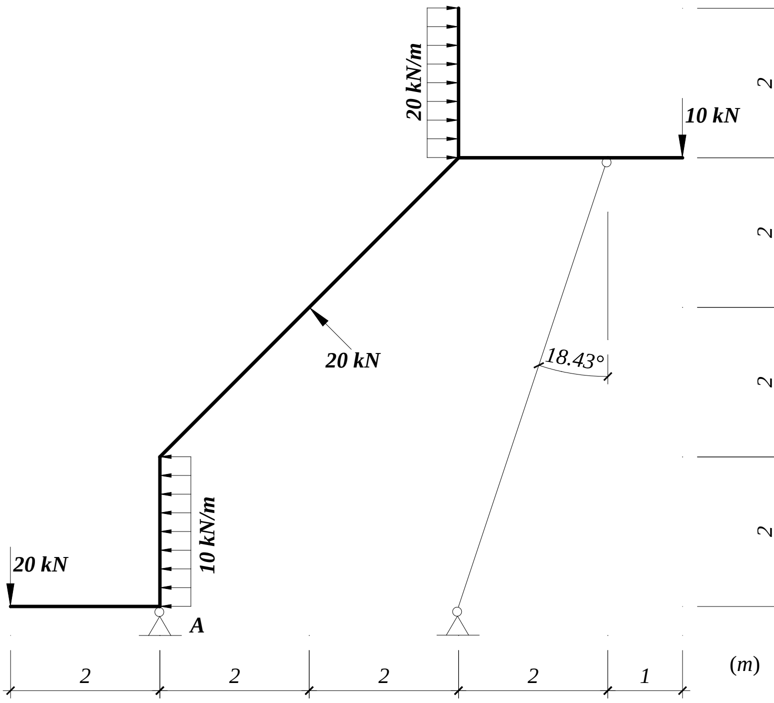




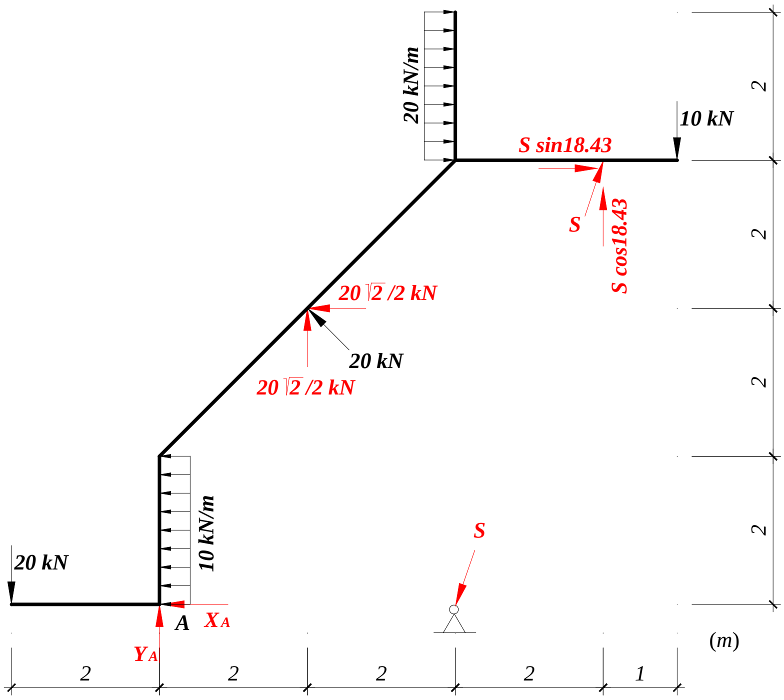

10
20
20
10
2.93
5.858
5
20
17.071
25.858
+
T (kN)


20
14.142
30
17.071
20
20
5
15
24.142
+
N (kN)

2. Za nosač na slici nacrtati dijagrame presječnih sila (M, T, N).



R J E Š E NJ E



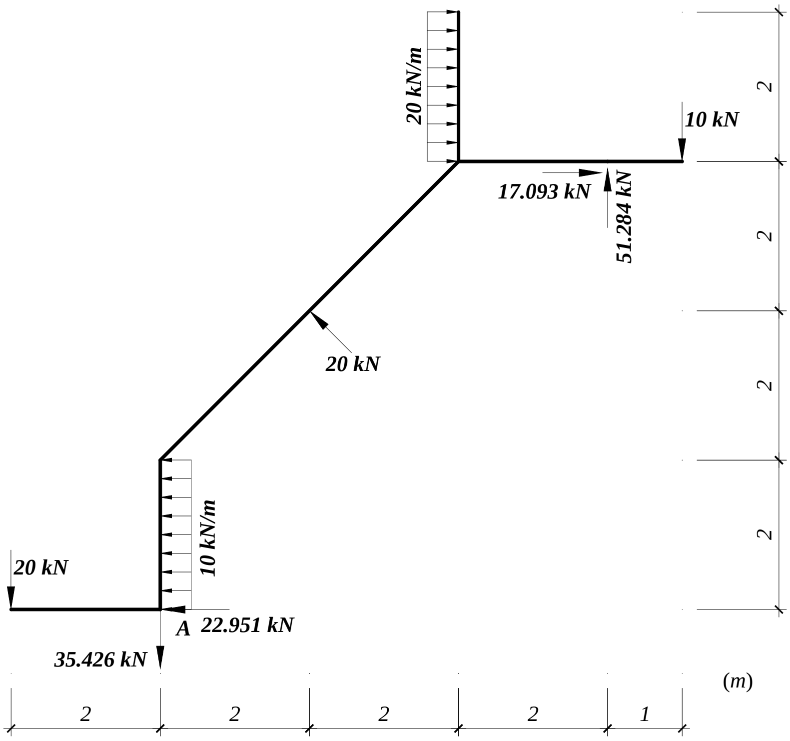
Uslovi ravnoteže:

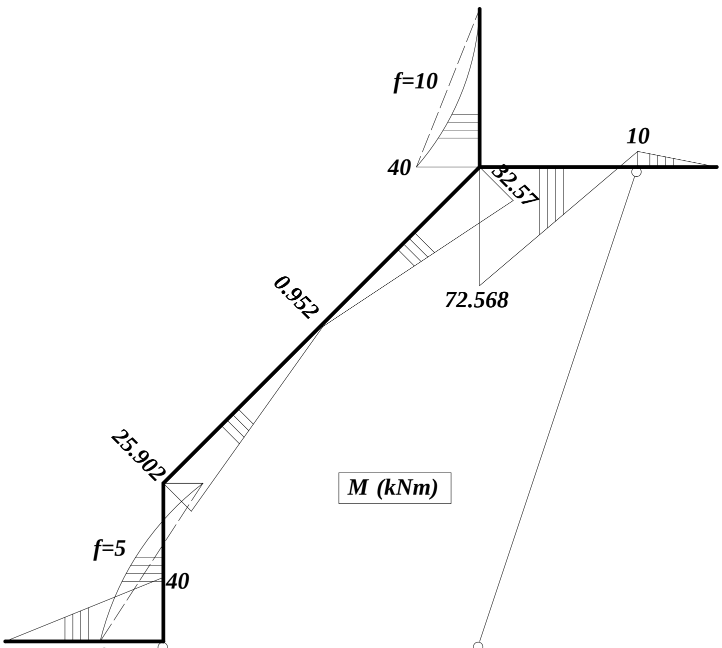
1. $\sum M_A = 0$ $\rightarrow$ $S\,cos18.43 \cdot 6 - S\,sin18.43 \cdot 6 - 20 \cdot 2 \cdot 7 + \frac{20\sqrt{2}}{2} \cdot (2+4) + 10 \cdot 2 \cdot 1 +$
$+ 20 \cdot 2 - 10 \cdot 7 = 0$ $\rightarrow$ $S = 54.057\ kN$

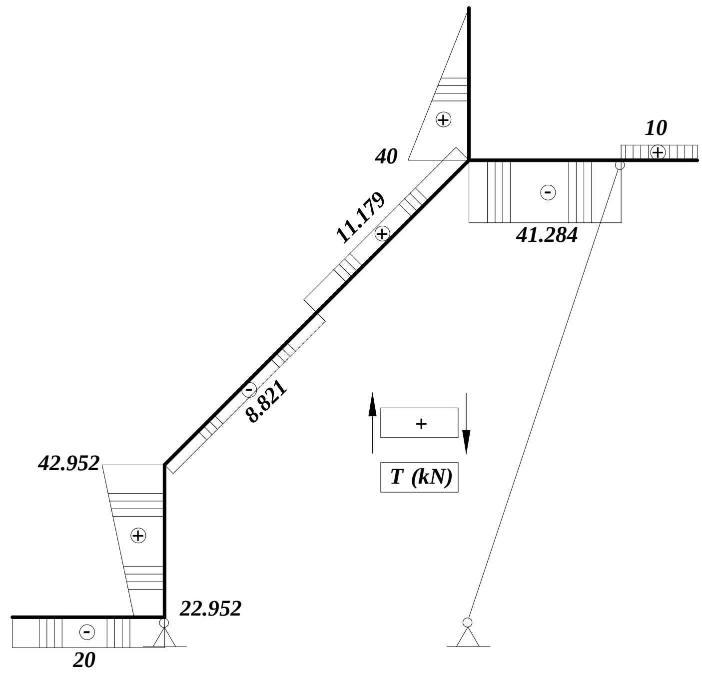
2. $\sum X = 0$ $\rightarrow$ $X_A + 10 \cdot 2 + \frac{20\sqrt{2}}{2} - 20 \cdot 2 - S\,sin18.43 = 0$ $\rightarrow$ $X_A = 22.95\ kN$

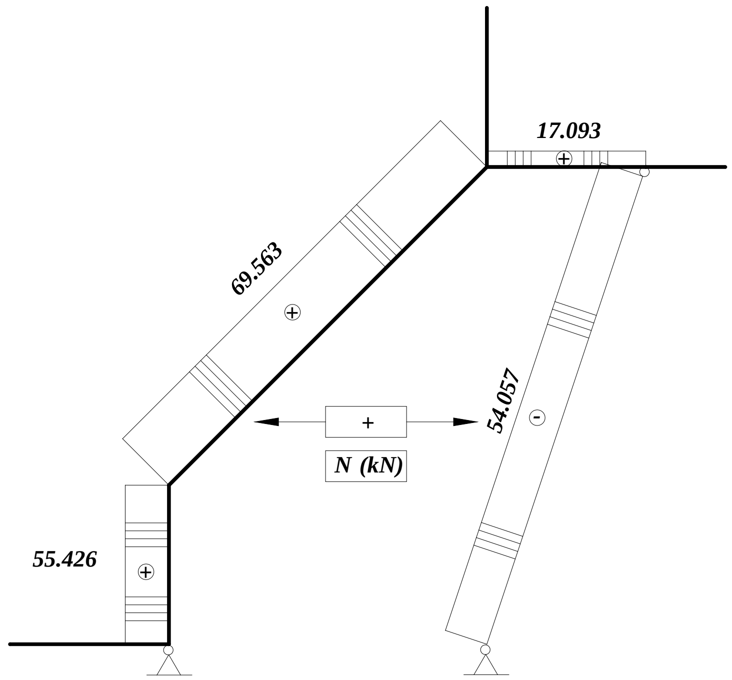
3. $\sum Y = 0$ $\rightarrow$ $Y_A - 20 + \frac{20\sqrt{2}}{2} - 10 + S\,cos18.43 = 0$ $\rightarrow$ $Y_A = -35.426\ kN$

Kontrola: npr. $\sum M_{(bilo\ koja\ tačka\ ravni)} = 0$ – zadovoljeno






10
40
11.179
41.284
8.821
+
T (kN)
42.952
22.952
20

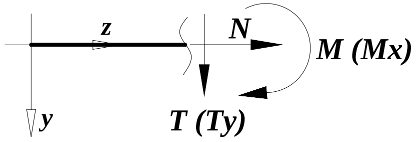

17.093
69.563
54.057
+
N (kN)
55.426

VJEŽBA BR. 2

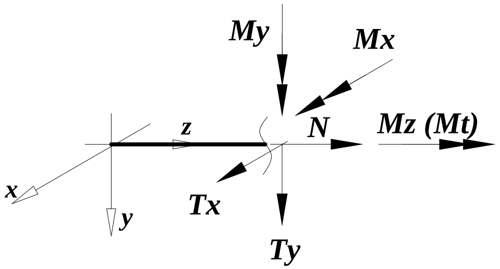
PONAVLJANJE GRADIVA IZ GRAĐEVINSKE MEHANIKE 1

PROSTORNI NOSAČI – IZLOMLJENE KONZOLE

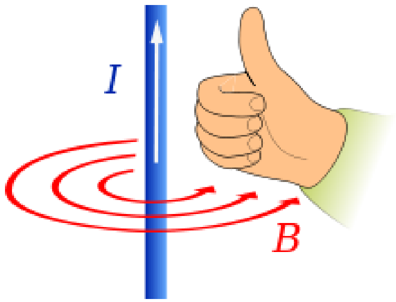
3 presjecne sile
nosaci u ravni



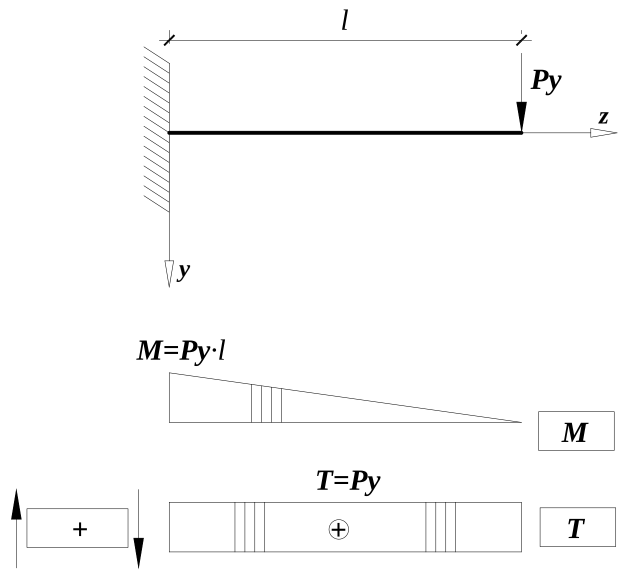
6 presjecnih sila
prostorni nosaci



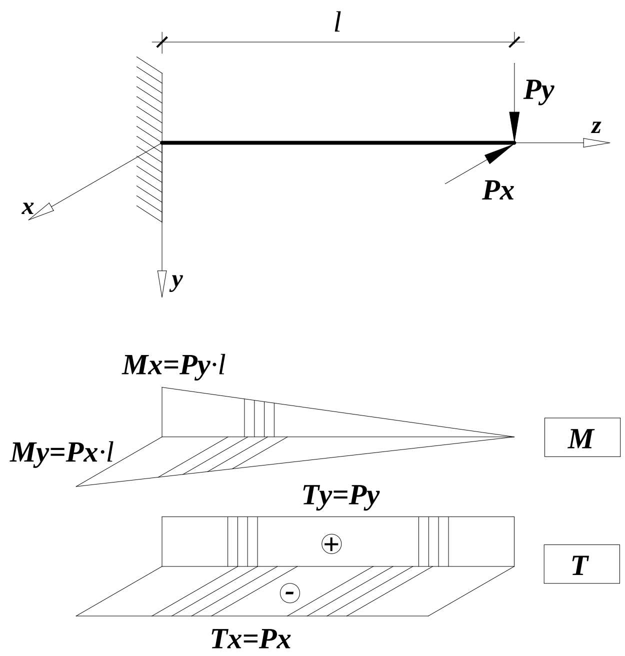
Pravilo
desne ruke



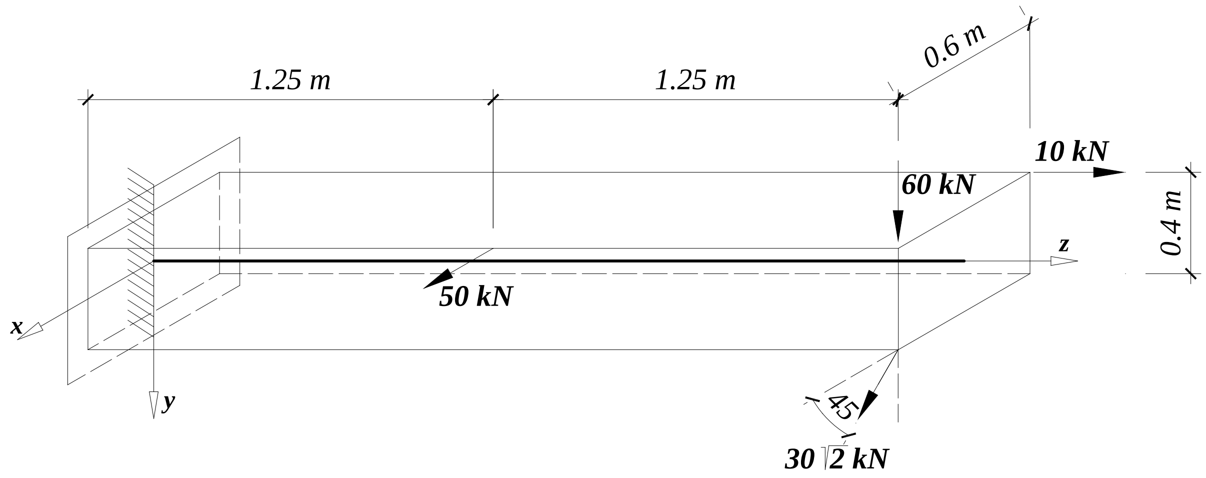
Nosac u ravni "yz"



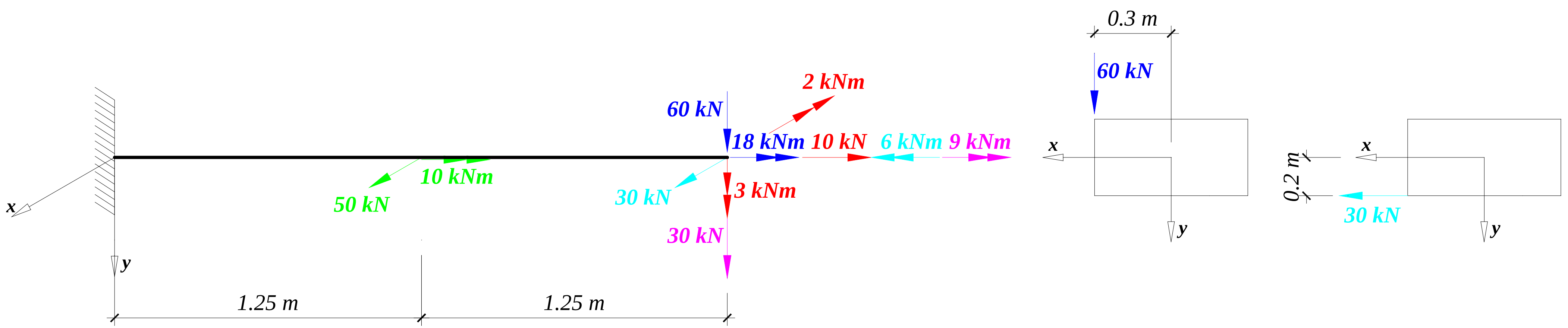
Prostorni nosac

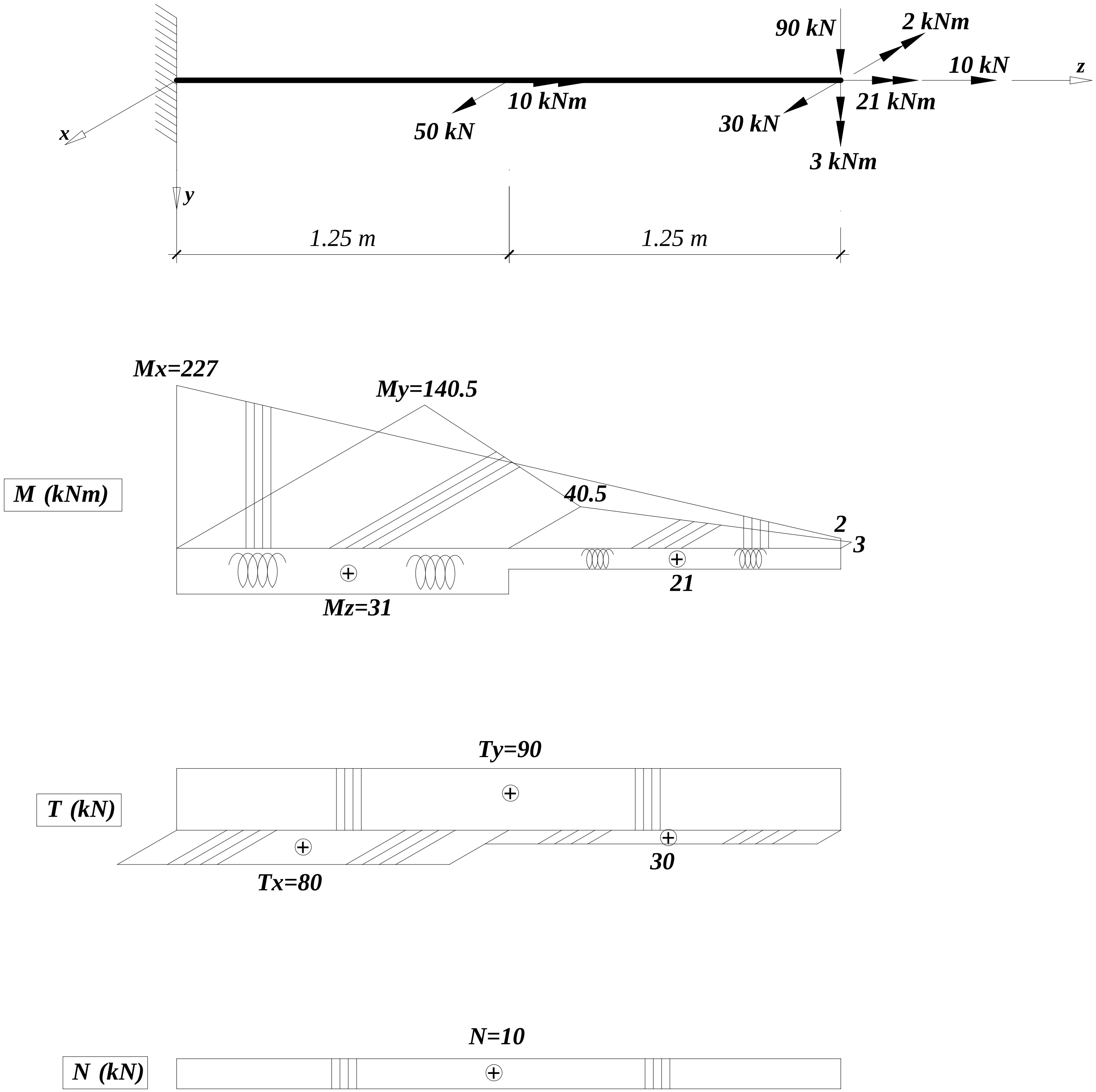


1. Za nosač na slici nacrtati dijagrame presječnih sila (M, T, N).



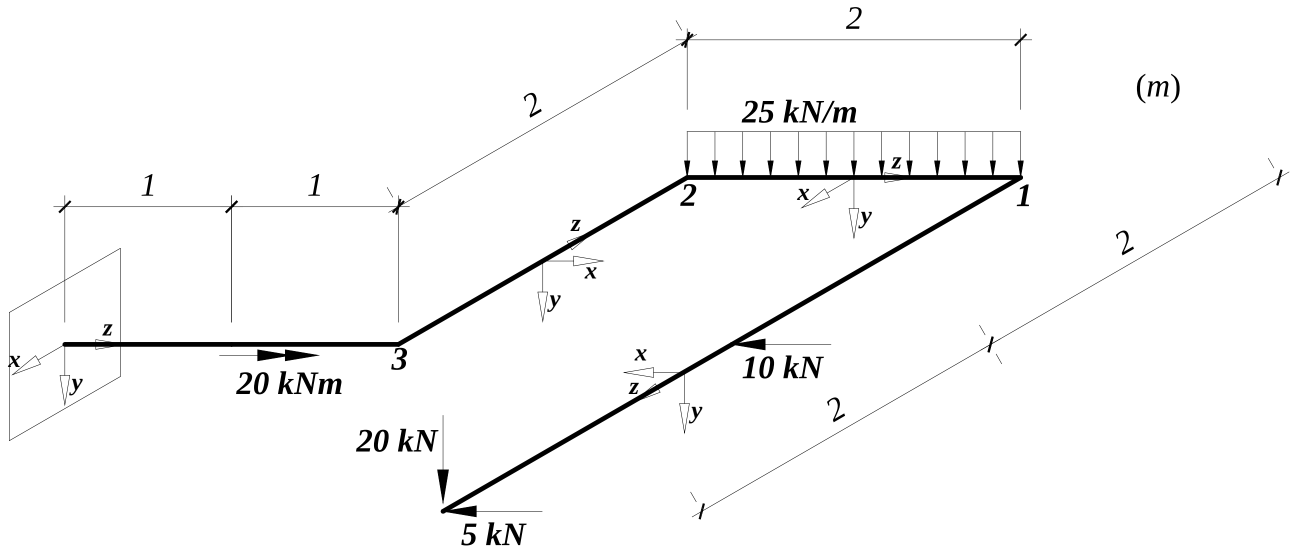
R J E Š E NJ E



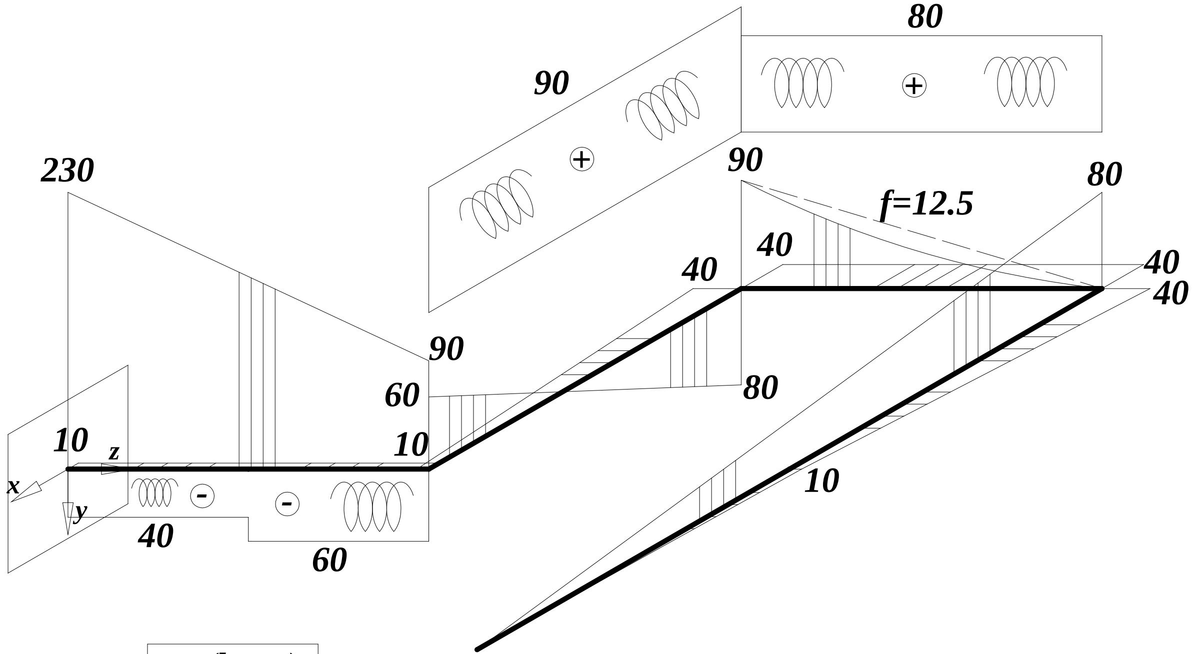


90 kN
2 kNm
10 kN
z
10 kNm
50 kN
30 kN
21 kNm
3 kNm
x
y
1.25 m
1.25 m
Mx=227
My=140.5
M (kNm)
40.5
2
3
21
Mz=31
Ty=90
T (kN)
30
Tx=80
N=10
N (kN)

2. Za nosač na slici nacrtati dijagrame presječnih sila (*M*, *T*, *N*).

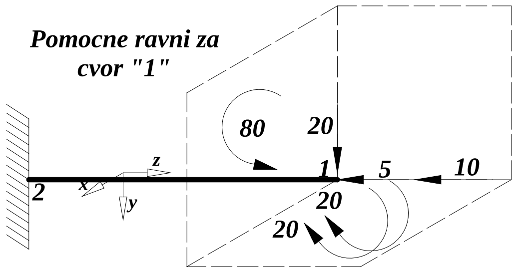


R J E Š E NJ E

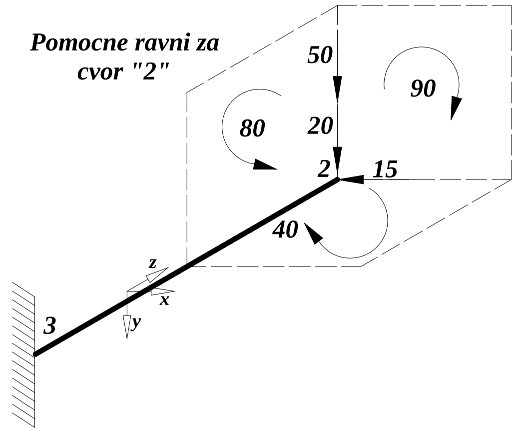


M (kNm)

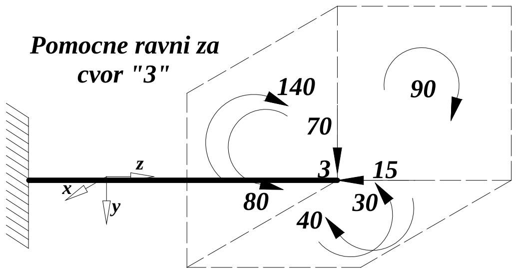
Pomocne ravni za cvor "1"

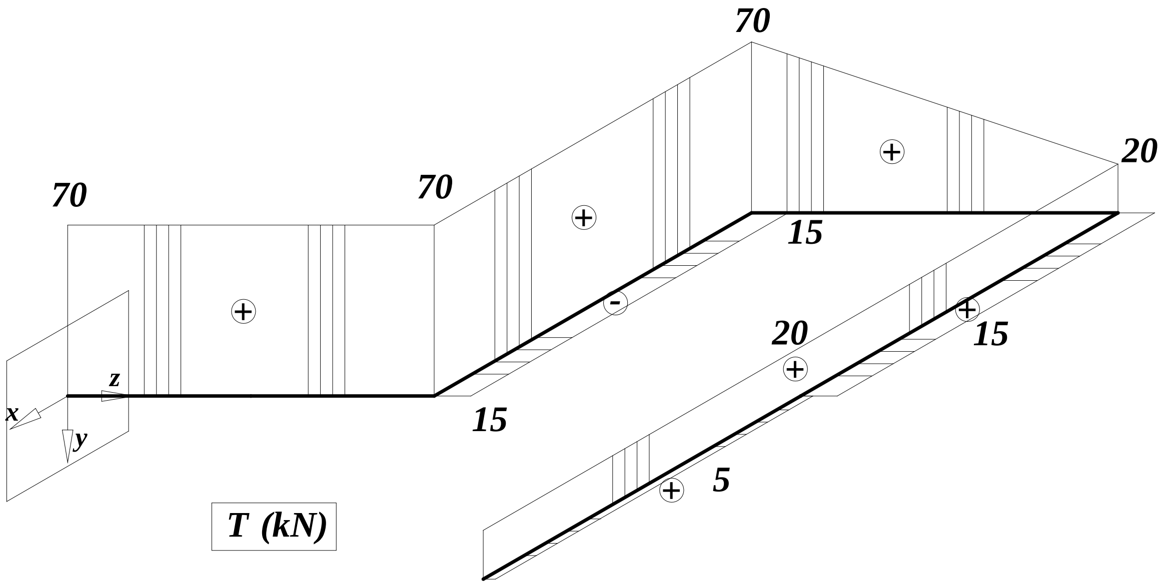


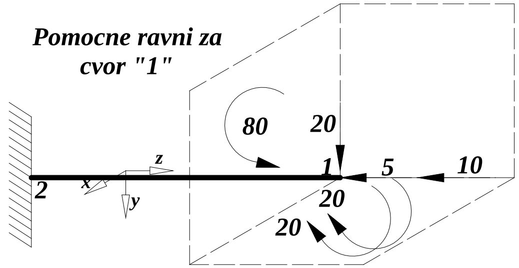
Pomocne ravni za cvor "2"



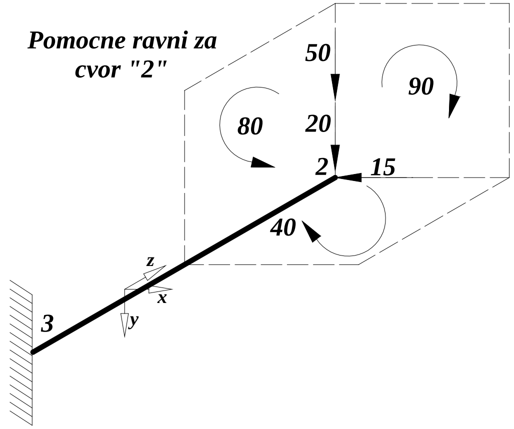
Pomocne ravni za cvor "3"



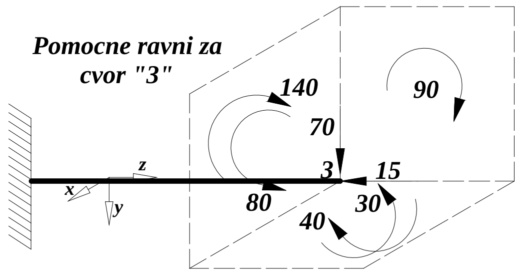




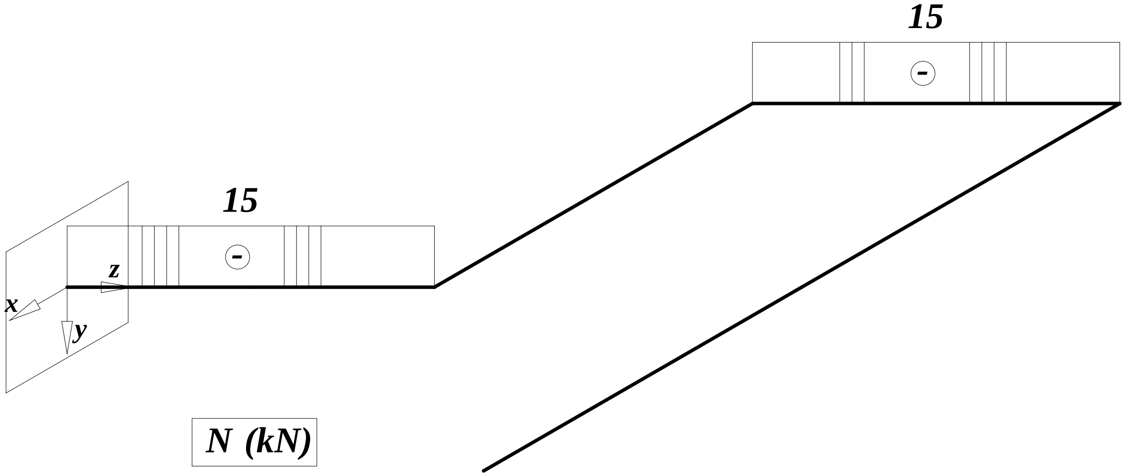
Pomocne ravni za cvor "1"



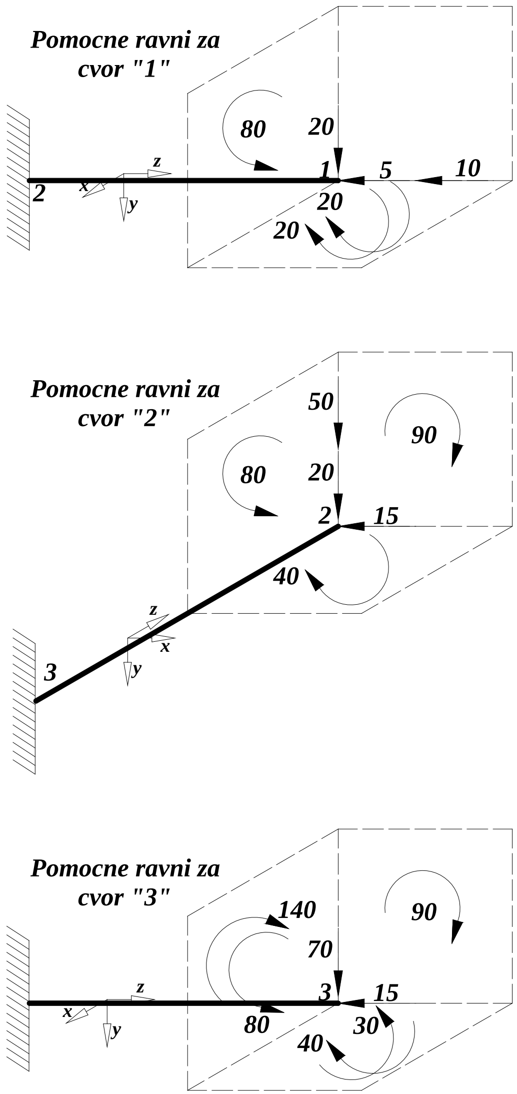
Pomocne ravni za cvor "2"



Pomocne ravni za cvor "3"

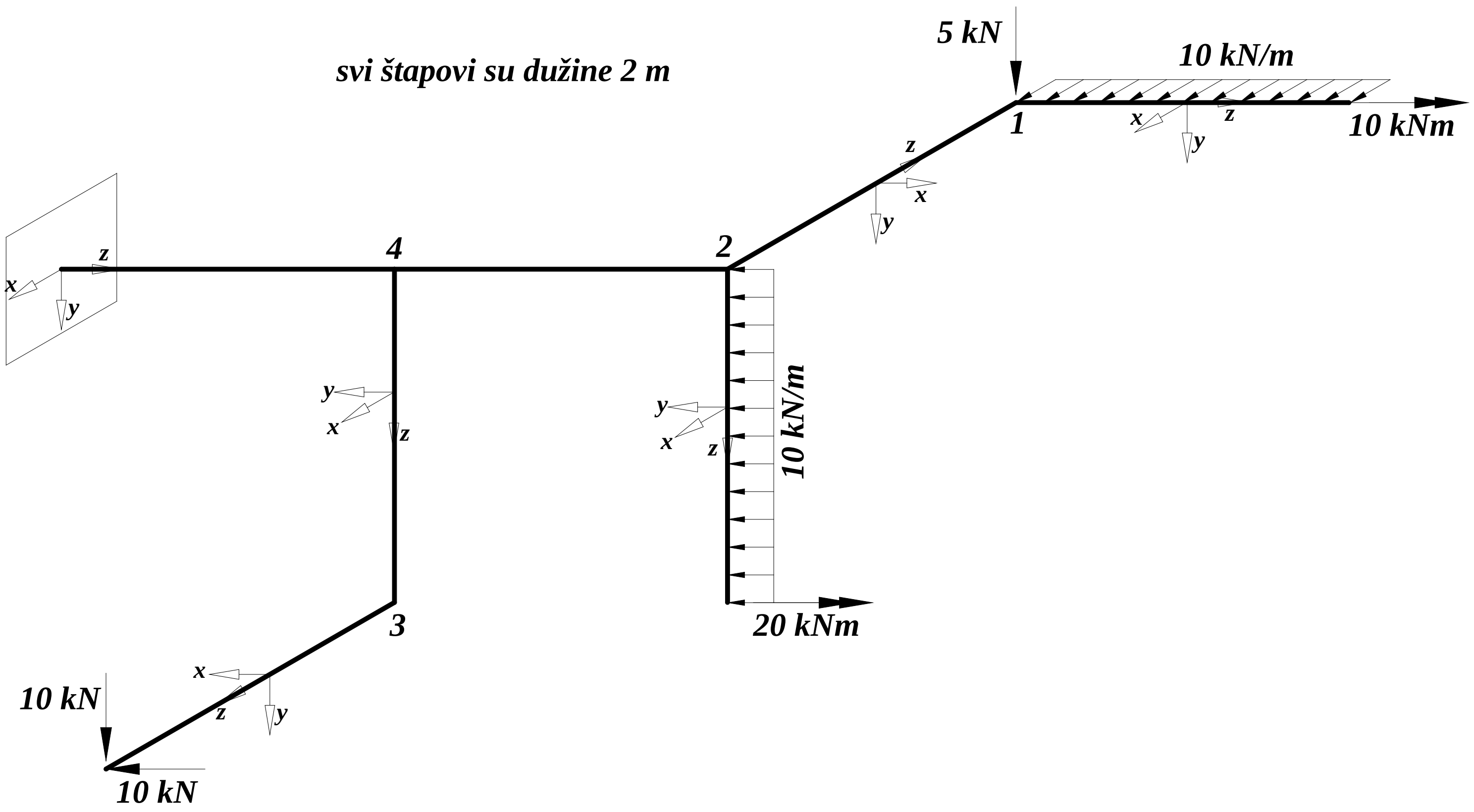


15
15
z
x
y
N (kN)

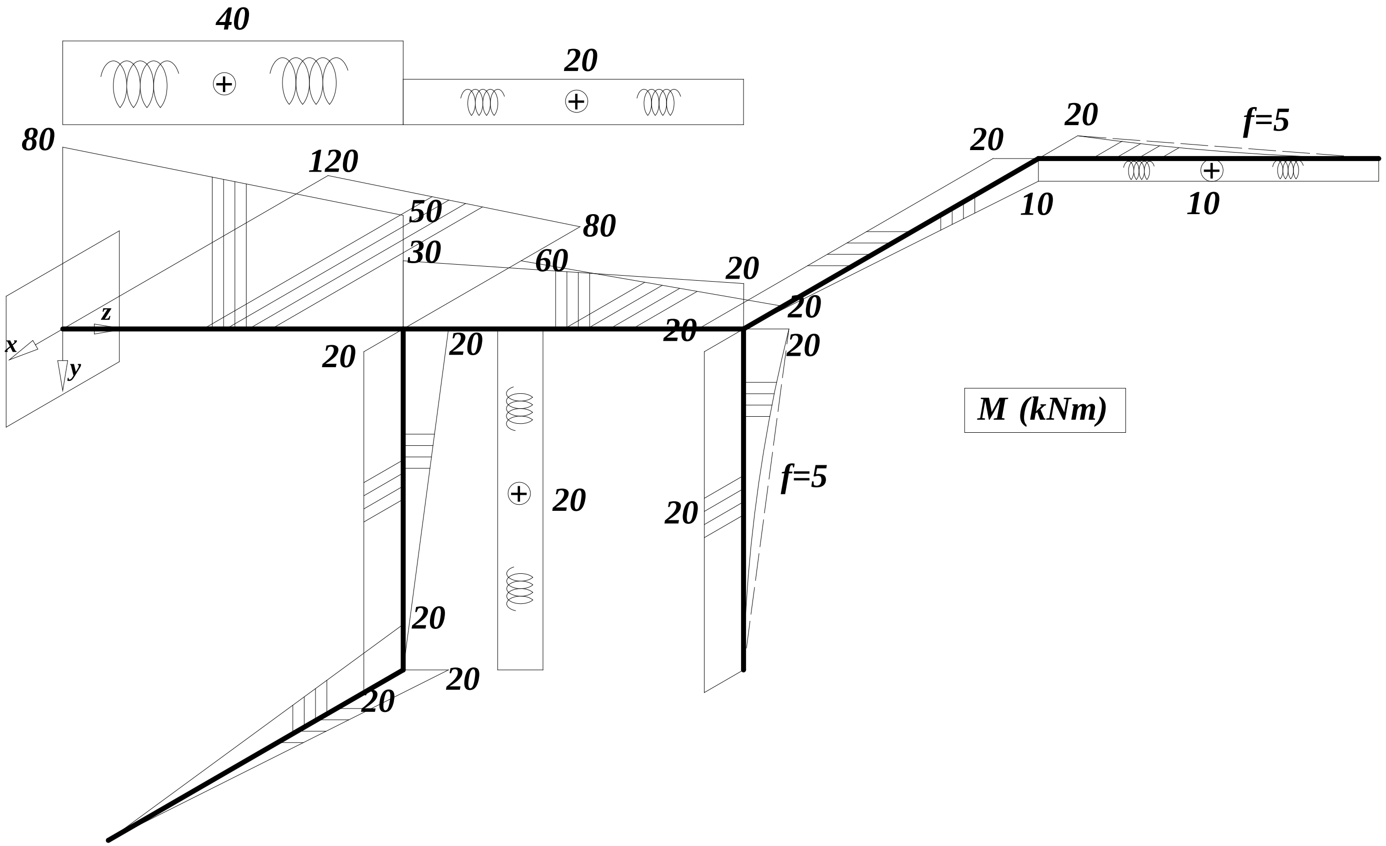


Pomocne ravni za cvor "1"
80
20
1
5
10
2
z
x
y
20
20
Pomocne ravni za cvor "2"
50
90
80
20
2
15
40
z
x
y
3
Pomocne ravni za cvor "3"
140
90
70
3
15
z
x
y
80
30
40

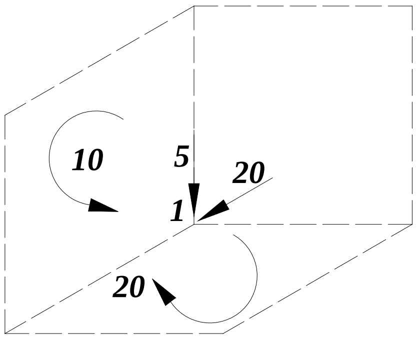
3. Za nosač na slici nacrtati dijagrame presječnih sila (M, T, N).



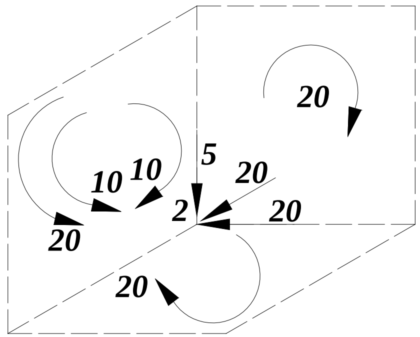
R J E Š E NJ E



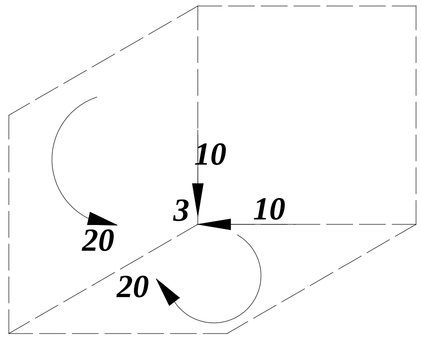
Pomocne ravni za cvor "1"



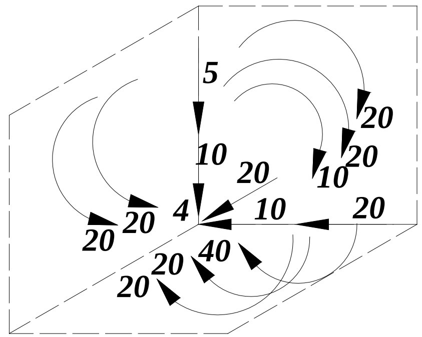
Pomocne ravni za cvor "2"

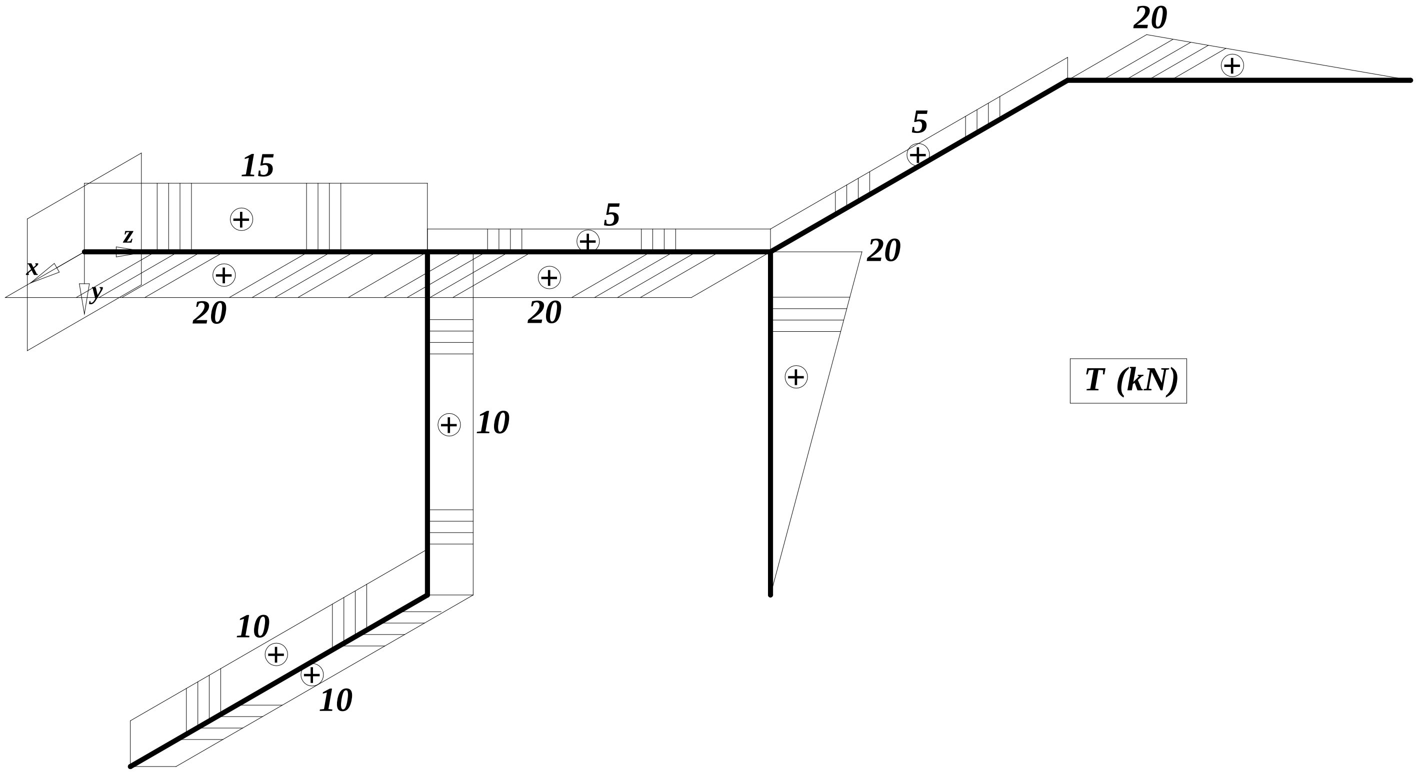


Pomocne ravni za cvor "3"

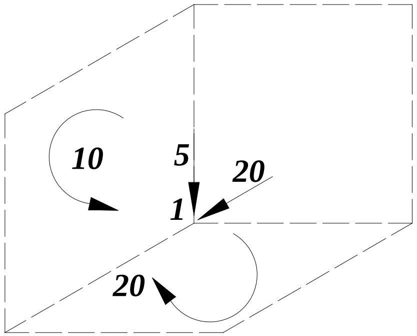


Pomocne ravni za cvor "4"

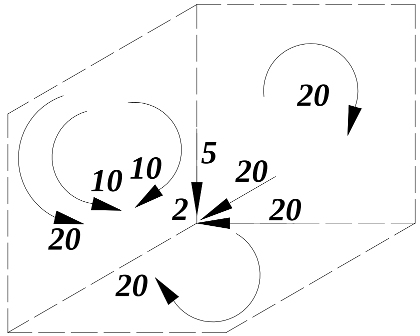




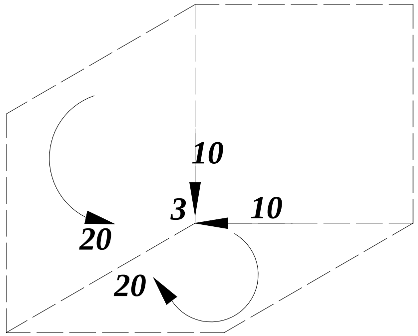
Pomocne ravni za cvor "1"



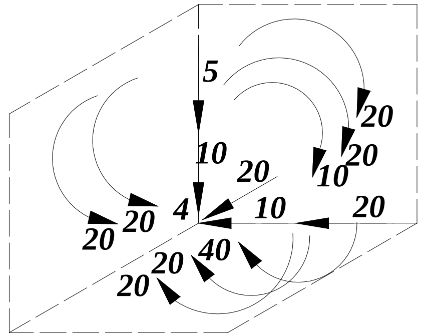
Pomocne ravni za cvor "2"

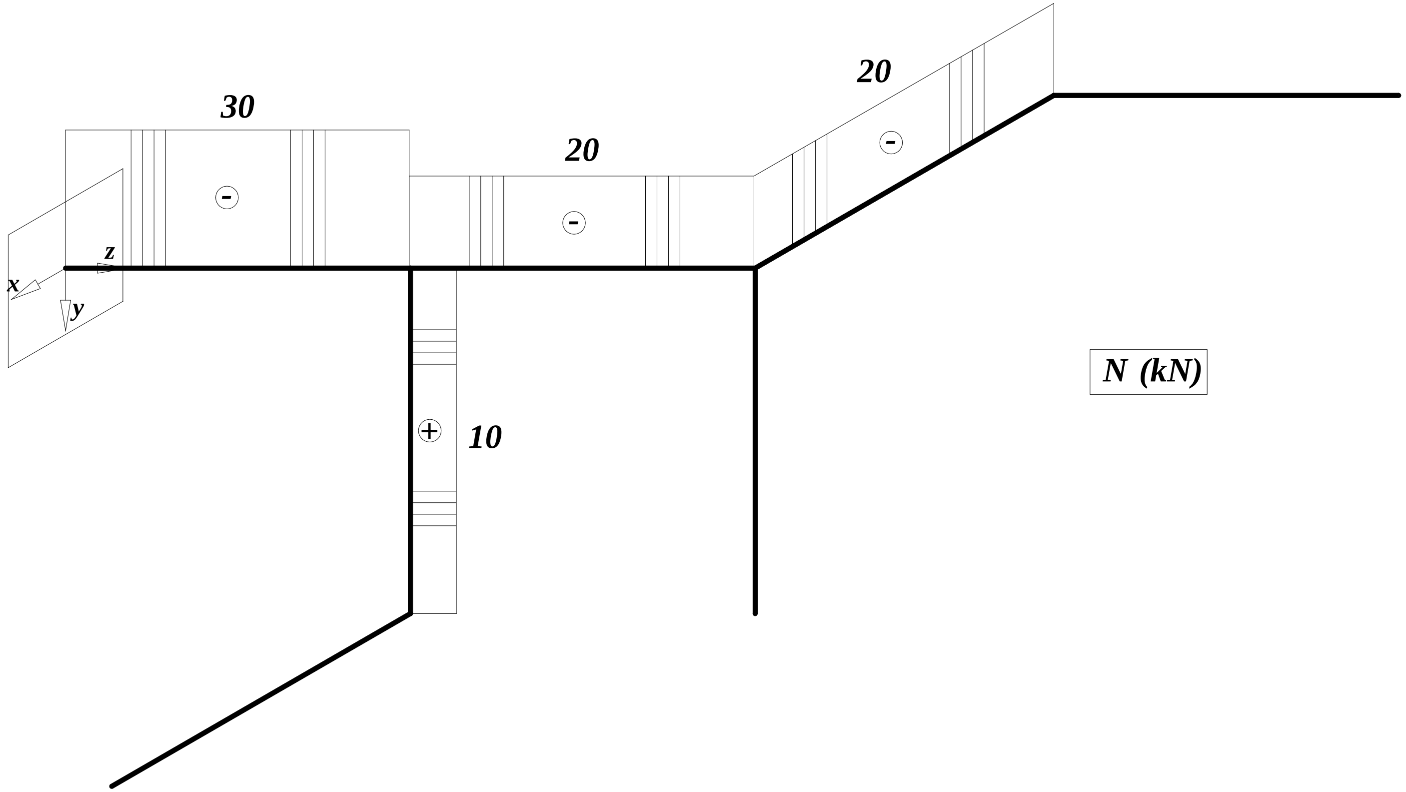


Pomocne ravni za cvor "3"

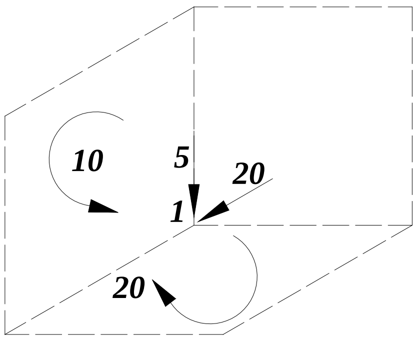


Pomocne ravni za cvor "4"

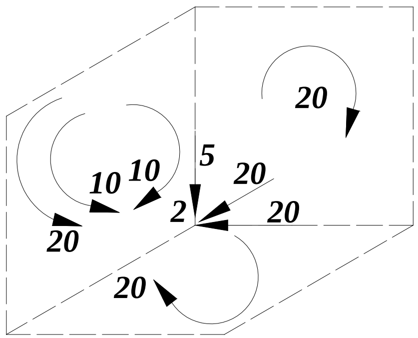




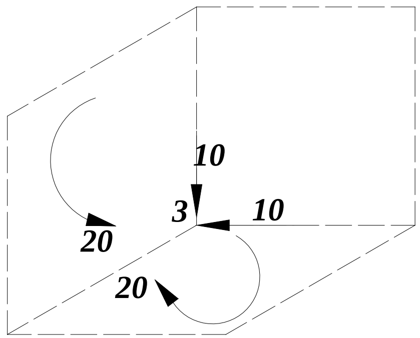
Pomocne ravni za cvor "1"



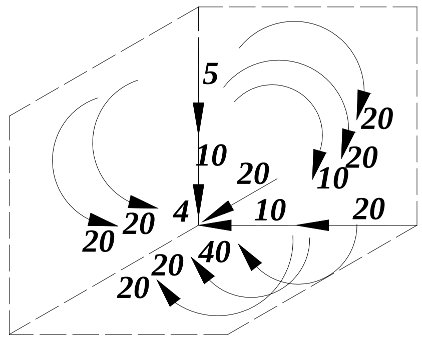
Pomocne ravni za cvor "2"



Pomocne ravni za cvor "3"



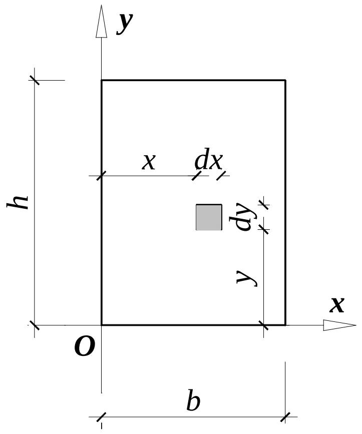
Pomocne ravni za cvor "4"



VJEŽBA BR. 3

GEOMETRIJSKE KARAKTERISTIKE RAVNIH POVRŠINA

1. Odrediti momente inercije u odnosu na *x* i *y* ose pravougaonika na slici.



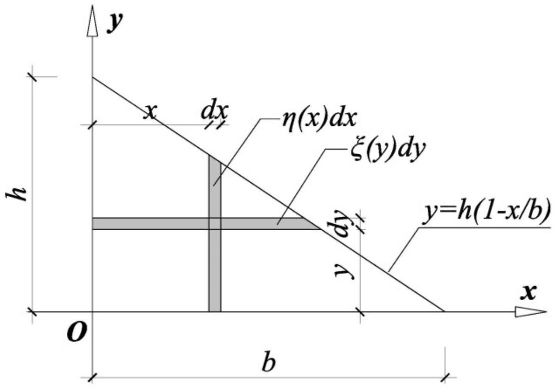
$$I_x = \int_A y^2 dA = \iint y^2 dxdy = \int_0^b dx \int_0^h y^2 dy = x\Big|_0^b \frac{y^3}{3}\Big|_0^h = \frac{bh^3}{3}$$

$$I_y = \int_A x^2 dA = \iint x^2 dxdy = \int_0^h dy \int_0^b x^2 dx = y\Big|_0^h \frac{x^3}{3}\Big|_0^b = \frac{hb^3}{3}$$

$$I_{xy} = \int_A xydA = \iint xydxdy = \int_0^b xdx \int_0^h ydy = \frac{x^2}{2}\Big|_0^b \frac{y^2}{2}\Big|_0^h = \frac{b^2h^2}{4}$$

$$I_O = I_x + I_y = \frac{bh^3}{3} + \frac{b^3h}{3} = \frac{bh}{3}\left(h^2 + b^2\right)$$

2. Odrediti momente inercije u odnosu na *x* i *y* ose pravouglog trougla na slici.



$$\xi(y) : b = (h-y) : h \quad \Rightarrow \quad \xi(y) = \frac{b}{h}(h-y)$$

$$\eta(x) : h = (b-x) : b \quad \Rightarrow \quad \eta(x) = \frac{h}{b}(b-x)$$

$$I_x = \int_A y^2 dA = \int_0^h y^2 \xi(y) dy = \int_0^h y^2 \frac{b}{h}(h-y)dy = \frac{b}{h}\int_0^h (y^2h - y^3)dy = \frac{b}{h}\left(\frac{y^3}{3}h - \frac{y^4}{4}\right)\Bigg|_0^h = \frac{b}{h}\left(\frac{h^4}{3} - \frac{h^4}{4}\right) = \frac{bh^3}{12}$$

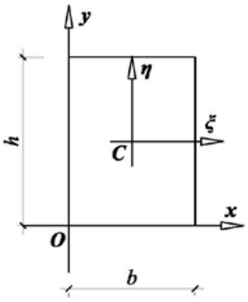
$$I_y = \int_A x^2 dA = \int_0^b x^2 \eta(x) dx = \int_0^b x^2 \frac{h}{b}(b-x)dx = \frac{h}{b}\int_0^b (x^2b - x^3)dx = \frac{h}{b}\left(\frac{x^3}{3}b - \frac{x^4}{4}\right)\Bigg|_0^b = \frac{h}{b}\left(\frac{b^4}{3} - \frac{b^4}{4}\right) = \frac{hb^3}{12}$$

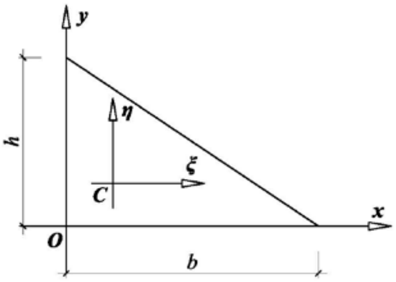
$$I_{xy} = \int_A xydA = \int_0^b xdx \int_0^{h(1-x/b)} ydy = \int_0^b x\frac{1}{2}h^2\left(1 - \frac{x}{b}\right)^2 dx = \frac{h^2}{2}\int_0^b \left(x - 2\frac{x^2}{b} + \frac{x^3}{b^2}\right)dx = \frac{h^2}{2}\left(\frac{x^2}{2} - 2\frac{x^3}{3b} + \frac{x^4}{4b^2}\right)\Bigg|_0^b =$$

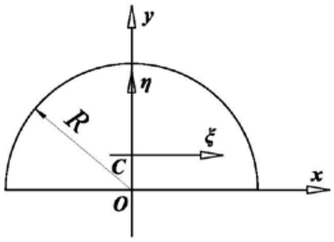
$$= \frac{h^2}{2}\left(\frac{b^2}{2} - \frac{2}{3}b^2 + \frac{1}{4}b^2\right) = \frac{h^2b^2}{2}\left(\frac{6-8+3}{12}\right) = \frac{b^2h^2}{24}$$

$$I_O = I_x + I_y = \frac{bh^3}{12} + \frac{b^3h}{12} = \frac{bh}{12}\left(h^2 + b^2\right)$$

3. Odrediti momente inercije figura na slici za sistem težišnih osa $\xi\eta$.

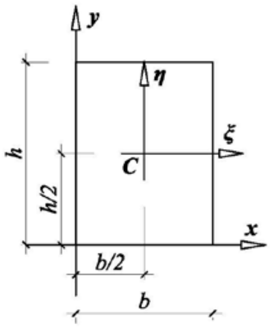




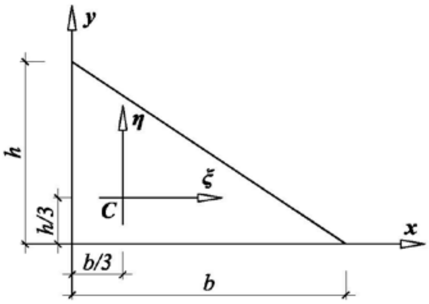


$I_x = \frac{bh^3}{3}$ $I_y = \frac{b^3h}{3}$ $I_{xy} = \frac{b^2h^2}{4}$ $I_x = \frac{bh^3}{12}$ $I_y = \frac{b^3h}{12}$ $I_{xy} = \frac{b^2h^2}{24}$ $I_x = \frac{\pi R^4}{8}$ $I_y = \frac{\pi R^4}{8}$ $I_{xy} = 0$

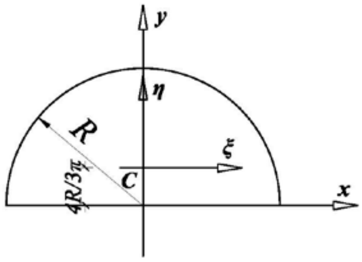
$$\boldsymbol{I_x = I_\xi + a^2A} \qquad \boldsymbol{I_\xi = I_x - a^2A}$$
$$\boldsymbol{I_y = I_\eta + b^2A} \qquad \textbf{\textit{Steiner}-ove jednačine} \qquad \boldsymbol{I_\eta = I_y - b^2A}$$
$$\boldsymbol{I_{xy} = I_{\xi\eta} + abA} \qquad \boldsymbol{I_{\xi\eta} = I_{xy} - abA}$$



$$I_\xi = \frac{bh^3}{3} - \left(\frac{h}{2}\right)^2 bh = \frac{bh^3}{12}$$
$$I_\eta = \frac{b^3h}{3} - \left(\frac{b}{2}\right)^2 bh = \frac{b^3h}{12}$$
$$I_{\xi\eta} = \frac{b^2h^2}{4} - \frac{h}{2}\frac{b}{2}bh = 0$$

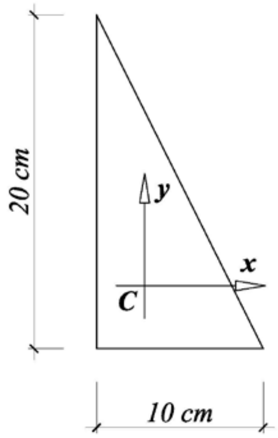


$$I_\xi = \frac{bh^3}{12} - \left(\frac{h}{3}\right)^2 \frac{1}{2}bh = \frac{bh^3}{36}$$
$$I_\eta = \frac{b^3h}{12} - \left(\frac{b}{3}\right)^2 \frac{1}{2}bh = \frac{b^3h}{36}$$
$$I_{\xi\eta} = \frac{b^2h^2}{24} - \frac{h}{3}\frac{b}{3}\frac{1}{2}bh = -\frac{b^2h^2}{72}$$



$$I_\xi = \frac{\pi R^4}{8} - \left(\frac{4R}{3\pi}\right)^2 \frac{1}{2}\pi R^2 = 0.10976R^4$$
$$I_\eta = \frac{\pi R^4}{8} - 0 = \frac{\pi R^4}{8}$$
$$I_{\xi\eta} = 0 - 0 = 0$$

4. Za pravougli trougao na slici odrediti glavne centralne ose inercije i glavne centralne momente inercije.



Centralni momenti inercije:

$$I_x = \frac{bh^3}{36} = \frac{10 \cdot 20^3}{36} = 2222.22\ cm^4$$
$$I_y = \frac{b^3 h}{36} = \frac{10^3 \cdot 20}{36} = 555.56\ cm^4$$
$$I_{xy} = -\frac{b^2 h^2}{72} = -\frac{10^2 \cdot 20^2}{72} = -555.56\ cm^4$$

$$I_1 = \frac{1}{2}\left(I_x + I_y\right) + \frac{1}{2}\sqrt{\left(I_x - I_y\right)^2 + 4I_{xy}^2}$$
$$I_2 = \frac{1}{2}\left(I_x + I_y\right) - \frac{1}{2}\sqrt{\left(I_x - I_y\right)^2 + 4I_{xy}^2}$$
$$\Rightarrow \quad I_{1/2} = \frac{1}{2}\left(I_x + I_y\right) \pm \frac{1}{2}\sqrt{\left(I_x - I_y\right)^2 + 4I_{xy}^2}$$

$$I_{1/2} = \frac{1}{2}\left(I_x + I_y\right) \pm \frac{1}{2}\sqrt{\left(I_x - I_y\right)^2 + 4I_{xy}^2} =$$
$$= \frac{1}{2}\left(2222.22 + 555.56\right) \pm \frac{1}{2}\sqrt{\left(2222.22 - 555.56\right)^2 + 4\left(-555.56\right)^2} =$$
$$= 1388.89 \pm 1001.54$$

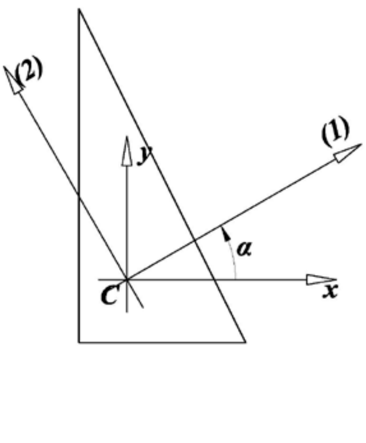
$\boldsymbol{I_1 = 2390.43\ cm^4}$
$\boldsymbol{I_2 = 387.35\ cm^4 > 0}$ Glavni centralni momenti inercije

Kontrola $I_x + I_y = I_1 + I_2$

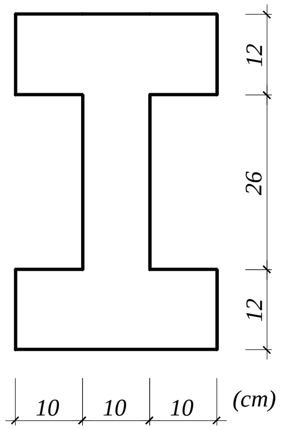
Položaj glavne centralne ose *(1)* u odnosu na horizontalnu osu (x) je definisan uglom (α):

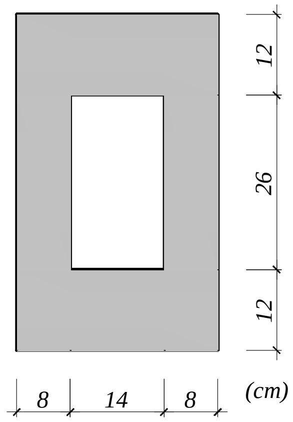
$$tg 2\alpha = \frac{-2I_{xy}}{I_x - I_y} = \frac{-2 \cdot (-555.56)}{2222.22 - 555.56} = 0.667$$

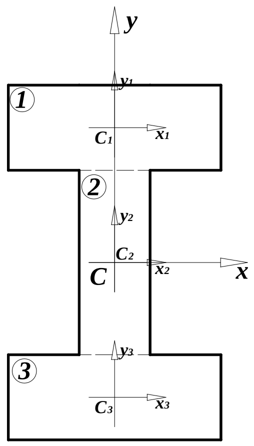
$$\boldsymbol{(I_x - I_y) > 0 \quad \alpha = \frac{1}{2} arctg 0.667 = 16.845^0}$$



5. Za presjeke na slici odrediti glavne centralne ose inercije i glavne centralne momente inercije.







$$\boldsymbol{I_x = I_x^{\boxed{1}} + I_x^{\boxed{2}} + I_x^{\boxed{3}} = 283\ 206.67\ cm^4 \ = \ I_1}$$

$$I_x^{\boxed{1}} = I_{x_1} + y_{C_1}{}^2 A_1 = \frac{30 \cdot 12^3}{12} + 19^2 \cdot 30 \cdot 12 = 134\ 280\ cm^4$$

$$I_x^{\boxed{2}} = I_{x_2} + y_{C_2}{}^2 A_2 = \frac{10 \cdot 26^3}{12} + 0 = 14\ 646.67\ cm^4$$

$$I_x^{\boxed{3}} = I_{x_3} + y_{C_3}{}^2 A_3 = \frac{30 \cdot 12^3}{12} + \left(-19\right)^2 \cdot 30 \cdot 12 = 134\ 280\ cm^4$$

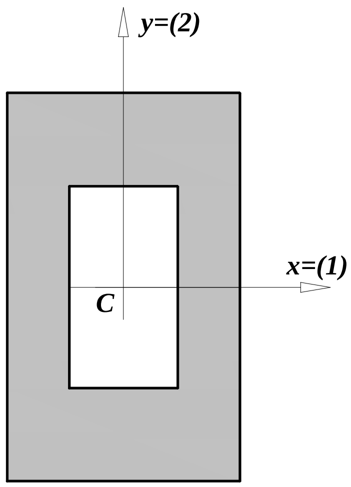
$$\boldsymbol{I_y = I_y^{\boxed{1}} + I_y^{\boxed{2}} + I_y^{\boxed{3}} = 56\ 166.67.67\ cm^4 \ = \ I_2}$$

$$I_y^{\boxed{1}} = I_{y_1} + x_{C_1}{}^2 A_1 = \frac{30^3 \cdot 12}{12} + 0 = 27\ 000\ cm^4 = I_y^{\boxed{3}}$$

$$I_y^{\boxed{2}} = I_{y_2} + x_{C_2}{}^2 A_2 = \frac{26 \cdot 10^3}{12} + 0 = 2\ 166.67\ cm^4$$

$I_{xy} = 0$ x i y su ose simetrije

$I_1 \equiv I_x \quad x \equiv (1)$
$I_2 \equiv I_y \quad y \equiv (2)$



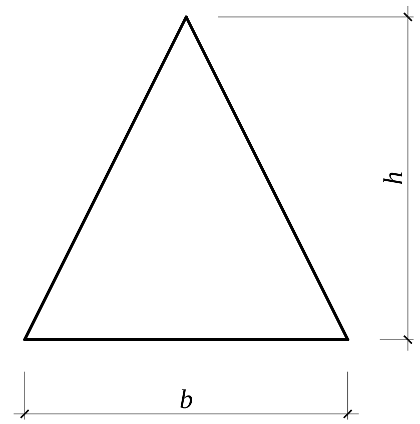
$I_x = 291\ 994.67\ cm^4 \ = \ I_1$

$I_y = 106\ 554.67\ cm^4 \ = \ I_2$

$I_{xy} = 0$ x i y su ose simetrije

$I_1 \equiv I_x \quad x \equiv (1)$
$I_2 \equiv I_y \quad y \equiv (2)$

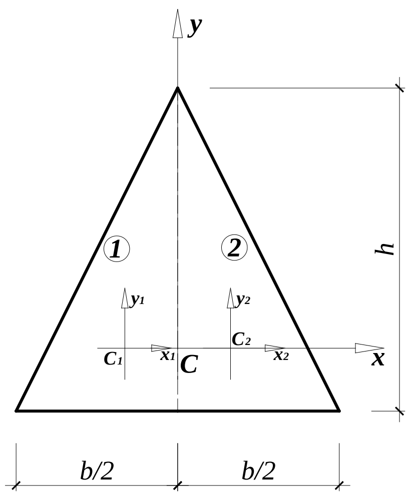
6. Za jednakokraki trougao na slici odrediti glavne centralne ose inercije i glavne centralne momente inercije.



$$I_x = I_x^{\boxed{1}} + I_x^{\boxed{2}} = \frac{\left(\frac{b}{2}\right)h^3}{36} + \frac{\left(\frac{b}{2}\right)h^3}{36} = \frac{bh^3}{36}$$

$$I_x^{\boxed{1}} = I_{x_1} + y_{C_1}^{\;2} A_1 = \frac{\left(\frac{b}{2}\right)h^3}{36} + 0$$

$$I_x^{\boxed{2}} = I_{x_2} + y_{C_2}^{\;2} A_2 = \frac{\left(\frac{b}{2}\right)h^3}{36} + 0$$



$$I_y = I_y^{\boxed{1}} + I_y^{\boxed{2}} = \frac{b^3h}{96} + \frac{b^3h}{96} = \frac{b^3h}{48}$$

$$I_y^{\boxed{1}} = I_{y_1} + x_{C_1}^{\;2} A_1 = \frac{\left(\frac{b}{2}\right)^3 h}{36} + \left(-\frac{b}{6}\right)^2 \frac{1}{2}\frac{b}{2}h = \frac{b^3h}{96}$$

$$I_y^{\boxed{2}} = I_{y_2} + x_{C_2}^{\;2} A_2 = \frac{\left(\frac{b}{2}\right)^3 h}{36} + \left(\frac{b}{6}\right)^2 \frac{1}{2}\frac{b}{2}h = \frac{b^3h}{96}$$

$$I_{xy} = I_{xy}^{\boxed{1}} + I_{xy}^{\boxed{2}} = 0$$

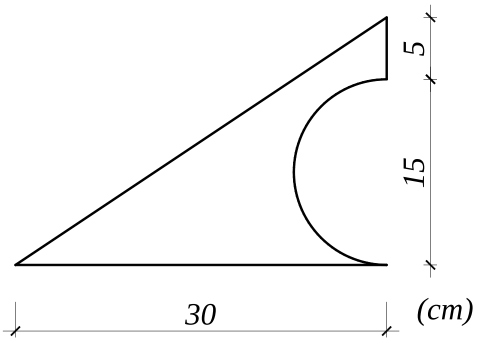
$$I_{xy}^{\boxed{1}} = I_{x_1y_1} + x_{C_1} y_{C_1} A_1 = \frac{\left(\frac{b}{2}\right)^2 h^2}{72} + 0$$

$$I_{xy}^{\boxed{2}} = I_{x_2y_2} + x_{C_2} y_{C_2} A_2 = -\frac{\left(\frac{b}{2}\right)^2 h^2}{72} + 0$$

VJEŽBA BR. 4

GEOMETRIJSKE KARAKTERISTIKE RAVNIH POVRŠINA

1. Za presjek na slici odrediti glavne centralne momente inercije i nacrtati glavne centralne ose inercije. Rezultat provjeriti *Mohr*-ovim krugom inercije.



R J E Š E NJ E

1. Težište

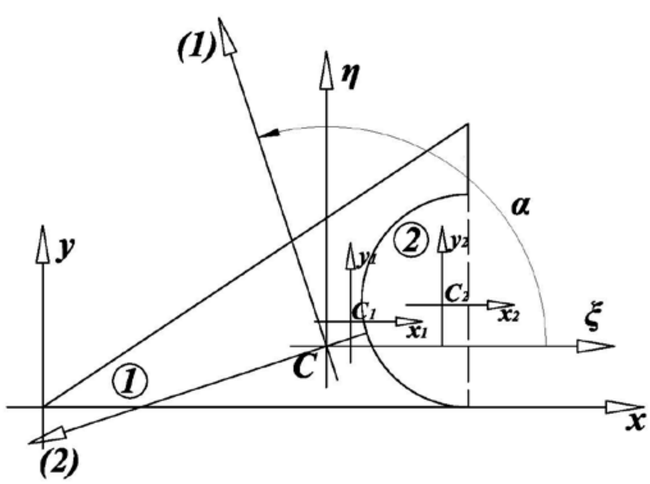
$C_1(x_{C_1}, y_{C_1}) = (20, 6.667)\,cm \quad A_1 = 300\ cm^2$

$C_2(x_{C_2}, y_{C_2}) = (26.817, 7.5)\,cm \quad A_2 = 88.357\ cm^2$

$$x_c = \frac{x_{c_1}A_1 - x_{c_2}A_2}{A_1 - A_2} = \frac{20 \cdot 300 - 26.817 \cdot 88.357}{300 - 88.357} = 17.154\ cm$$

$$y_c = \frac{y_{c_1}A_1 - y_{c_2}A_2}{A_1 - A_2} = \frac{6.667 \cdot 300 - 7.5 \cdot 88.357}{300 - 88.357} = 6.319\ cm$$

$\boldsymbol{C(x_C, y_C) = (17.154,\ 6.319)\,cm}$

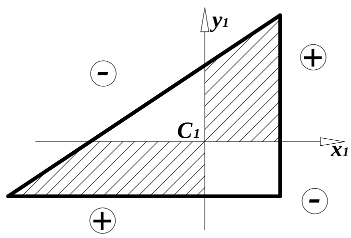


2. Centralni momenti inercije

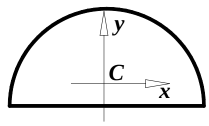
$\boldsymbol{I_\xi = I_\xi^{\boxed{1}} - I_\xi^{\boxed{2}} = 5\ 337.239\ cm^4}$

$$I_\xi^{\boxed{1}} = I_{x_1} + (y_{C1} - y_C)^2 A_1 = \frac{30 \cdot 20^3}{36} + (6.667 - 6.319)^2\, 300 = 6\ 703\ cm^4$$

$$I_\xi^{\boxed{2}} = I_{x_2} + (y_{C2} - y_C)^2 A_2 = \frac{7.5^4 \pi}{8} + (7.5 - 6.319)^2\, 88.357 = 1\ 365.761\ cm^4$$

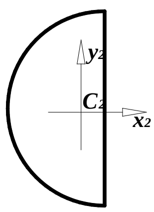


$\boldsymbol{I_\eta = I_\eta^{\boxed{1}} - I_\eta^{\boxed{2}} = 8\ 832.419\ cm^4}$



$$I_\eta^{\boxed{1}} = I_{y_1} + (x_{C1} - x_C)^2 A_1 = \frac{30^3 \cdot 20}{36} + (20 - 17.154)^2\, 300 = 17\ 429.915\ cm^4$$

$$I_\eta^{\boxed{2}} = I_{y_2} + (x_{C2} - x_C)^2 A_2 = 0.10976 R^4 + (26.817 - 17.154)^2\, 88.357 = 8\ 597.496\ cm^4$$



$\boldsymbol{I_{\xi\eta} = I_{\xi\eta}^{\boxed{1}} - I_{\xi\eta}^{\boxed{2}} = 4\ 288.792\ cm^4}$

$$I_{\xi\eta}^{\boxed{1}} = I_{x_1 y_1} + (y_{C1} - y_C)(x_{C1} - x_C) A_1 = +\frac{30^2 \cdot 20^2}{72} + 0.348 \cdot 2.846 \cdot 300 = 5\ 297.122\ cm^4$$

$$I_{\xi\eta}^{\boxed{2}} = I_{x_2 y_2} + \left(y_{C2} - y_C\right)\left(x_{C2} - x_C\right) A_1 = 0 + 9.663 \cdot 1.181 \cdot 88.357 = 1\,008.330\ cm^4$$

3. Glavni centralni momenti inercije

$$I_{1/2} = \frac{1}{2}\left(I_\xi + I_\eta\right) \pm \frac{1}{2}\sqrt{\left(I_\xi - I_\eta\right)^2 + 4I_{\xi\eta}^2} =$$
$$= \frac{1}{2}\left(5\,337.239 + 8\,832.419\right) \pm \frac{1}{2}\sqrt{\left(5\,337.239 - 8\,832.419\right)^2 + 4\left(4\,288.792\right)^2} =$$
$$= 7\,084.829 \pm 4\,631.178$$

$\boldsymbol{I_1 = 11\,716.007\ cm^4}$

$\boldsymbol{I_2 = 2\,453.651\ cm^4 > 0}$

Kontrola $I_\xi + I_\eta = I_1 + I_2$

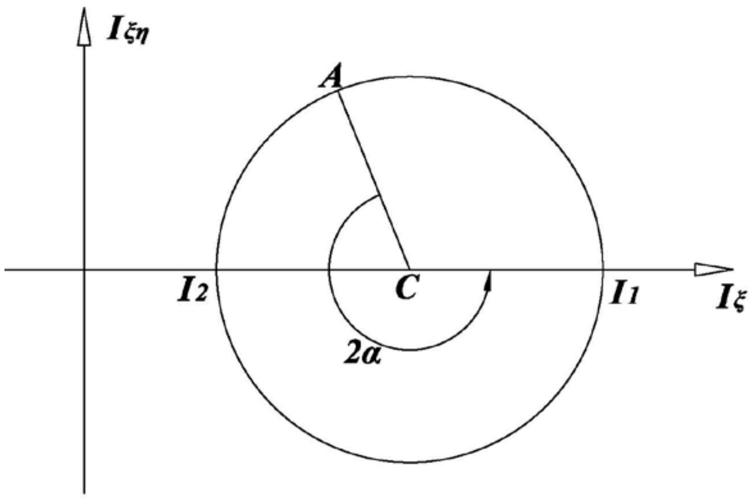
$$tg2\alpha = \frac{-2I_{\xi\eta}}{I_\xi - I_\eta} = \frac{-2 \cdot 4\,288.792}{5\,337.239 - 8\,832.419} = 2.454$$

$$\boldsymbol{(I_\xi - I_\eta) < 0 \quad \alpha = \frac{1}{2} arctg 2.454 + 90^0 = 123.91^0}$$

4. *Mohr*-ov krug inercije

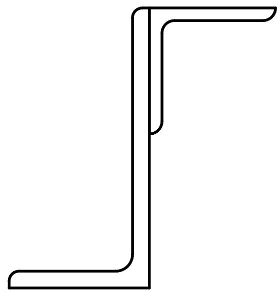
$$C\left[\frac{1}{2}\left(I_\xi + I_\eta\right);0\right] = \left[\frac{1}{2}\left(5\,337.239 + 8\,832.419\right);\ 0\right] = \left[7\,084.829;\ 0\right]\ cm^4$$
$$A\left[I_\xi; I_{\xi\eta}\right] = \left[5\,337.239;\ 4\,288.792\right]\ cm^4$$

Usvaja se razmjera 1 000 cm^4 = 1*cm*



2. Za presjek na slici odrediti glavne centralne momente inercije i nacrtati glavne centralne ose inercije. Rezultat provjeriti Mohr-ovim krugom inercije.

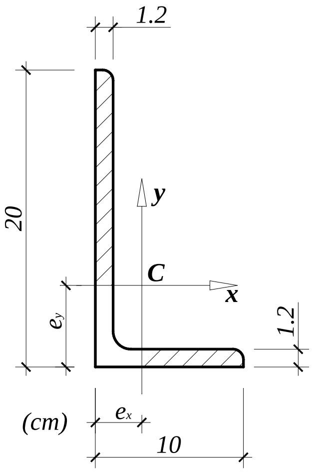
L 90x90x9



L 100x200x12

R J E Š E NJ E

L 100x200x12



$e_x = 2.10\,cm$

$e_y = 7.03\,cm$

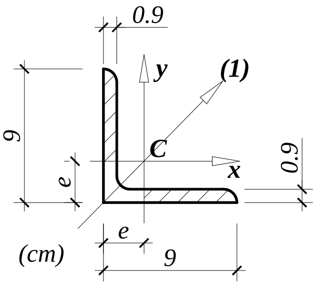
$A = 34.8\,cm^2$

$I_x = 1440\,cm^4$

$I_y = 247\,cm^4$

$I_{xy} = -338\,cm^4$

L 90x90x9



$e_x = 2.54\,cm$

$A = 15.5\,cm^2$

$I_x = I_y = 116\,cm^4$

$I_1 = 184\,cm^4$

$\left|I_{xy}\right| = I_1 - I_x = 184 - 116 = 68\,cm^4$

$I_{xy} = -68\ cm^4$

1. Težište

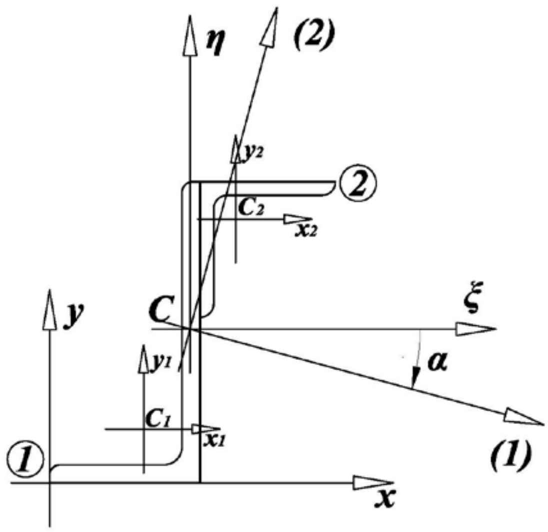
$C_1(x_{C_1}, y_{C_1}) = (10 - 2.10,\ 7.03) = (7.9,\ 7.03)\,cm \quad A_1 = 34.8\ cm^2$

$C_2(x_{C_2}, y_{C_2}) = (10 + 2.54,\ 20 - 2.54) = (12.54,\ 17.46)\,cm \quad A_2 = 15.5$

$$x_c = \frac{x_{c_1}A_1 + x_{c_2}A_2}{A_1 + A_2} = \frac{7.9\cdot 34.8 + 12.54\cdot 15.5}{34.8 + 15.5} = 9.33\ cm$$

$$y_c = \frac{y_{c_1}A_1 + y_{c_2}A_2}{A_1 + A_2} = \frac{7.03\cdot 34.8 + 17.46\cdot 15.5}{34.8 + 15.5} = 10.24\ cm$$

$\boldsymbol{C(x_C, y_C) = (9.33,\ 10.24)\,cm}$



2. Centralni momenti inercije

$\boldsymbol{I_\xi = I_\xi^{\boxed{1}} + I_\xi^{\boxed{2}} = 2\,722.57\ cm^4}$

$$I_\xi^{\boxed{1}} = I_{x_1} + \left(y_{C1} - y_C\right)^2 A_1 = 1440 + \left(7.03 - 10.24\right)^2 34.8 = 1\,798.58\ cm^4$$

$$I_\xi^{\boxed{2}} = I_{x_2} + \left(y_{C2} - y_C\right)^2 A_2 = 116 + \left(17.46 - 10.24\right)^2 15.5 = 923.99\ cm^4$$

$\boldsymbol{I_\eta = I_\eta^{\boxed{1}} + I_\eta^{\boxed{2}} = 593.87\ cm^4}$

$$I_\eta^{\boxed{1}} = I_{y_1} + \left(x_{C1} - x_C\right)^2 A_1 = 247 + \left(7.9 - 9.33\right)^2 34.8 = 318.16\ cm^4$$

$$I_{\eta}^{\boxed{2}} = I_{y_2} + \left(x_{C2} - x_C\right)^2 A_2 = 116 + \left(12.54 - 9.33\right)^2 15.5 = 275.71\; cm^4$$

$$\boldsymbol{I_{\xi\eta} = I_{\xi\eta}^{\boxed{1}} + I_{\xi\eta}^{\boxed{2}} = 924.97\; cm^4}$$

$$I_{\xi\eta}^{\boxed{1}} = I_{x_1 y_1} + \left(y_{C1} - y_C\right)\left(x_{C1} - x_C\right) A_1 = +338 + \left(-1.43\right)\cdot\left(-3.21\right)\cdot 34.8 = 497.74\; cm^4$$

$$I_{\xi\eta}^{\boxed{2}} = I_{x_2 y_2} + \left(y_{C2} - y_C\right)\left(x_{C2} - x_C\right) A_1 = 68 + 3.21\cdot 7.22\cdot 15.5 = 427.23\; cm^4$$

3. Glavni centralni momenti inercije

$$I_{1/2} = \frac{1}{2}\left(I_{\xi} + I_{\eta}\right) \pm \frac{1}{2}\sqrt{\left(I_{\xi} - I_{\eta}\right)^2 + 4I_{\xi\eta}^2} =$$
$$= \frac{1}{2}\left(2722.57 + 593.87\right) \pm \frac{1}{2}\sqrt{\left(2722.57 - 593.87\right)^2 + 4\left(924.97\right)^2} =$$
$$= 1\,658.22 \pm 1\,410.11$$

$$\boldsymbol{I_1 = 3\,068.33\; cm^4}$$
$$\boldsymbol{I_2 = 248.11\; cm^4 > 0}$$

Kontrola $I_{\xi} + I_{\eta} = I_1 + I_2$

$$tg2\alpha = \frac{-2I_{\xi\eta}}{I_{\xi} - I_{\eta}} = \frac{-2\cdot 924.97}{2\,722.57 - 593.87} = -0.869$$

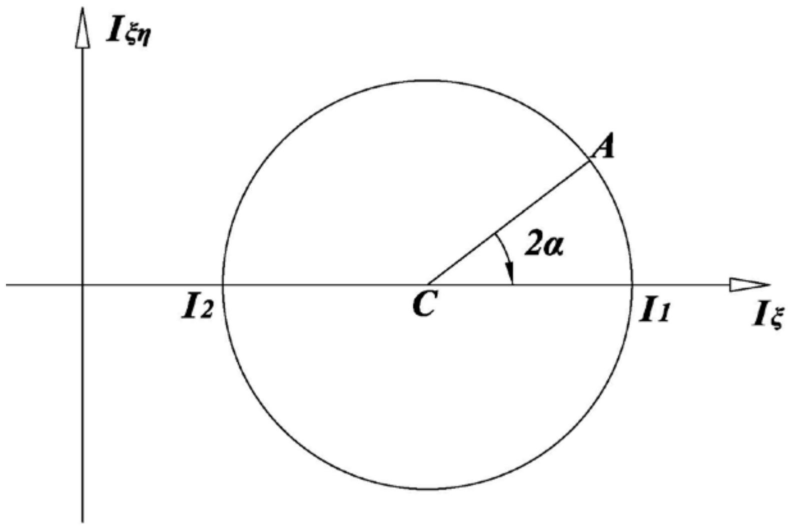
$$\boldsymbol{(I_{\xi} - I_{\eta}) > 0 \quad \alpha = \frac{1}{2} arctg(-0.869) = -20.5^0}$$

4. *Mohr*-ov krug inercije

$$C\left[\frac{1}{2}\left(I_{\xi} + I_{\eta}\right); 0\right] = \left[\frac{1}{2}\left(2\,722.57 + 593.87\right);\; 0\right] = \left[1\,658.22;\; 0\right]\; cm^4$$

$$A\left[I_{\xi}; I_{\xi\eta}\right] = \left[2\,722.57;\; 924.97\right]\; cm^4$$

Usvaja se razmjera 1 000 cm^4 = 1*cm*



VJEŽBA BR. 5

ANALIZA NAPONA

1. Stanje napona u tački opterećenog tijela zadato je tenzorom napona [S].
   a. Pokazati da je stanje napona linijsko;
   b. Odrediti pravac sa kojim su paralelni vektori svih totalnih napona;
   c. Napisati tenzor napona u sistemu glavnih osa.

$$[S]=\begin{bmatrix}3 & 1 & 2\\ 1 & 1/3 & 2/3\\ 2 & 2/3 & 4/3\end{bmatrix}\ (Pa)$$

R J E Š E NJ E

a. Linijsko stanje napona: $\overrightarrow{p_x}=\lambda\overrightarrow{p_n}$, $\overrightarrow{p_z}=\mu\overrightarrow{p_n}$ ili $\det[S]=0$, $Minori=0$

$$[S]=\begin{bmatrix}3 & 1 & 2\\ 1 & 1/3 & 2/3\\ 2 & 2/3 & 4/3\end{bmatrix} \rightarrow \left.\begin{array}{l}\overrightarrow{p_x}=3\vec{i}+\vec{j}+2\vec{k}\\ \overrightarrow{p_y}=\vec{i}+\frac{1}{3}\vec{j}+\frac{2}{3}\vec{k}\\ \overrightarrow{p_z}=2\vec{i}+\frac{2}{3}\vec{j}+\frac{4}{3}\vec{k}\end{array}\right\} \rightarrow \left.\begin{array}{l}\overrightarrow{p_x}=3\overrightarrow{p_y}\\ \overrightarrow{p_z}=2\overrightarrow{p_y}\end{array}\right\} \rightarrow \text{linijsko stanje napona, ili}$$

$$\left.\begin{array}{l}\det[S]=\begin{vmatrix}3 & 1 & 2\\ 1 & 1/3 & 2/3\\ 2 & 2/3 & 4/3\end{vmatrix}=3\begin{vmatrix}1/3 & 2/3\\ 2/3 & 4/3\end{vmatrix}-1\begin{vmatrix}1 & 2/3\\ 2 & 4/3\end{vmatrix}+2\begin{vmatrix}1 & 1/3\\ 2 & 2/3\end{vmatrix}=\\ =3\left(\frac{4}{9}-\frac{4}{9}\right)-1\left(\frac{4}{3}-\frac{4}{3}\right)+2\left(\frac{2}{3}-\frac{2}{3}\right)=0\end{array}\right\} \rightarrow \det[S]$$

$$\left.\begin{array}{lll}\begin{vmatrix}1/3 & 2/3\\ 2/3 & 4/3\end{vmatrix}=0 & \begin{vmatrix}3 & 1\\ 1 & 1/3\end{vmatrix}=0 & \begin{vmatrix}3 & 1\\ 2 & 2/3\end{vmatrix}=0\\ \begin{vmatrix}1 & 2/3\\ 2 & 4/3\end{vmatrix}=0 & \begin{vmatrix}3 & 2\\ 1 & 2/3\end{vmatrix}=0 & \begin{vmatrix}3 & 2\\ 2 & 4/3\end{vmatrix}=0\\ \begin{vmatrix}1 & 1/3\\ 2 & 2/3\end{vmatrix}=0 & \begin{vmatrix}1 & 2\\ 1/3 & 2/3\end{vmatrix}=0 & \begin{vmatrix}1 & 2\\ 2/3 & 4/3\end{vmatrix}=0\end{array}\right\} \rightarrow Minori=0$$

$\det[S]$ i $Minori=0$ $\rightarrow$ linijsko stanje napona

b. Linijsko stanje napona $\rightarrow$ vektori totalnog napona za bilo koju presječnu ravan u okolini tačke imaju isti pravac. Neka je to pravac $\vec{n}=\cos\alpha\vec{i}+\cos\beta\vec{j}+\cos\gamma\vec{k}$

Posmatrajmo npr. vektor totalnog napona za ravan sa normalom x

$$\overrightarrow{p_x}=\sigma_x\vec{i}+\tau_{xy}\vec{j}+\tau_{xz}\vec{k}=3\vec{i}+\vec{j}+2\vec{k}$$

$$\cos\alpha=\frac{\sigma_x}{\left|\overrightarrow{p_x}\right|}=\frac{3}{\sqrt{3^2+1^2+2^2}}=\frac{3}{\sqrt{14}}$$

$$\cos\beta = \frac{\tau_{xy}}{\left|\overrightarrow{p_x}\right|} = \frac{1}{\sqrt{3^2+1^2+2^2}} = \frac{1}{\sqrt{14}}$$

$$\cos\gamma = \frac{\tau_{xz}}{\left|\overrightarrow{p_x}\right|} = \frac{2}{\sqrt{3^2+1^2+2^2}} = \frac{2}{\sqrt{14}}$$

$$\boxed{\vec{n} = \frac{3}{\sqrt{14}}\vec{i} + \frac{1}{\sqrt{14}}\vec{j} + \frac{2}{\sqrt{14}}\vec{k}}$$ - pravac sa kojim su paralelni vektori svih totalnih napona

$$\overrightarrow{p_n} = p_{nx}\vec{i} + p_{ny}\vec{j} + p_{nz}\vec{k}$$

$$p_{nx} = \overrightarrow{p_n}\cdot\vec{i} = \overrightarrow{p_x}\cdot\vec{n} = (3\vec{i}+\vec{j}+2\vec{k})\cdot\left(\frac{3}{\sqrt{14}}\vec{i}+\frac{1}{\sqrt{14}}\vec{j}+\frac{2}{\sqrt{14}}\vec{k}\right) = 3\frac{3}{\sqrt{14}}+\frac{1}{\sqrt{14}}+2\frac{2}{\sqrt{14}} = 3.741$$

$$p_{ny} = \overrightarrow{p_n}\cdot\vec{j} = \overrightarrow{p_y}\cdot\vec{n} = (\vec{i}+\frac{1}{3}\vec{j}+\frac{2}{3}\vec{k})\cdot\left(\frac{3}{\sqrt{14}}\vec{i}+\frac{1}{\sqrt{14}}\vec{j}+\frac{2}{\sqrt{14}}\vec{k}\right) = \frac{3}{\sqrt{14}}+\frac{1}{3}\frac{1}{\sqrt{14}}+\frac{2}{3}\frac{2}{\sqrt{14}} = 1.247$$

$$p_{nz} = \overrightarrow{p_n}\cdot\vec{k} = \overrightarrow{p_z}\cdot\vec{n} = (2\vec{i}+\frac{2}{3}\vec{j}+\frac{4}{3}\vec{k})\cdot\left(\frac{3}{\sqrt{14}}\vec{i}+\frac{1}{\sqrt{14}}\vec{j}+\frac{2}{\sqrt{14}}\vec{k}\right) = 2\frac{3}{\sqrt{14}}+\frac{2}{3}\frac{1}{\sqrt{14}}+\frac{4}{3}\frac{2}{\sqrt{14}} = 1.247$$

$$\overrightarrow{p_n} = 3.741\vec{i}+1.247\vec{j}+2.494\vec{k}$$

$$\left|\overrightarrow{p_n}\right| = \sqrt{3.741^2+1.247^2+2.494^2} = 4.666\ Pa$$

$$\sigma_n = \overrightarrow{p_n}\cdot\vec{n} = \left(3.741\vec{i}+1.247\vec{j}+2.494\vec{k}\right)\cdot\left(\frac{3}{\sqrt{14}}\vec{i}+\frac{1}{\sqrt{14}}\vec{j}+\frac{2}{\sqrt{14}}\vec{k}\right) = 3.741\frac{3}{\sqrt{14}}+1.247\frac{1}{\sqrt{14}}+2.494\frac{2}{\sqrt{14}} = 4.666\,Pa$$

c. Tenzor napona u sistemu glavnih osa

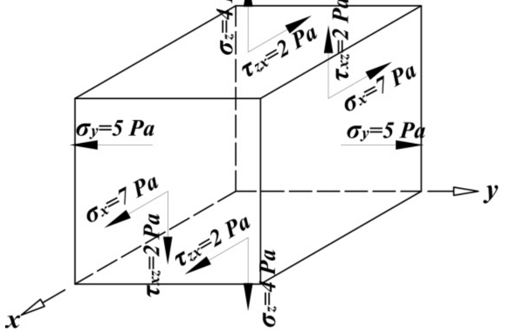
$$\left[S\right] = \begin{bmatrix} 4.666 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\ (Pa)$$

2. Stanje napona u tački opterećenog tijela zadato je tenzorom napona $[S]$.
   a. Prikazati okolinu tačke;
   b. Odrediti totalni napon za ravan sa normalom $\vec{n} = \left\{\frac{2}{3}; -\frac{2}{3}; \frac{1}{3}\right\}$;
   c. Za totalni napon iz tačke *c*. odrediti normalni napon i totalni smičući napon.

$$\left[S\right] = \begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{bmatrix}\ (Pa)$$

R J E Š E NJ E

a. Okolina tačke



b. Totalni napon za ravan sa normalom $\vec{n}$

$$[S]=\begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{bmatrix} \rightarrow \left.\begin{matrix} \overrightarrow{p_x}=7\vec{i}-2\vec{k} \\ \overrightarrow{p_y}=5\vec{j} \\ \overrightarrow{p_z}=-2\vec{i}+4\vec{k} \end{matrix}\right\}$$ - Vektori totalnog napona za 3 međusobno upravne ravni

$$\vec{n}=\left\{\frac{2}{3};-\frac{2}{3};\frac{1}{3}\right\}=\frac{2}{3}\vec{i}-\frac{2}{3}\vec{j}+\frac{1}{3}\vec{k}$$

$$|\vec{n}|=\sqrt{\left(\frac{2}{3}\right)^2+\left(-\frac{2}{3}\right)^2+\left(\frac{1}{3}\right)^2}=1$$ u slučaju da nijesmo dobili 1 $\rightarrow \vec{n}=\cos\alpha\vec{i}+\cos\beta\vec{j}+\cos\gamma\vec{k}$

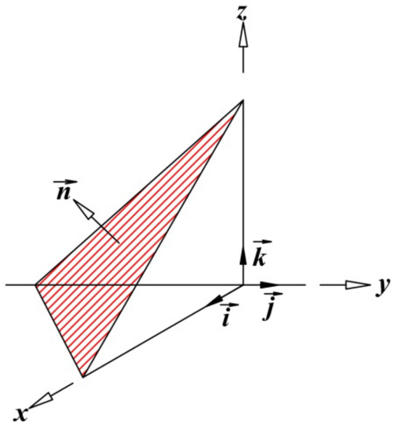
$$\cos\alpha=\frac{n_x}{|\vec{n}|}=\frac{2/3}{1};\ \cos\beta=\frac{n_y}{|\vec{n}|}=\frac{-2/3}{1};\ \cos\gamma=\frac{n_z}{|\vec{n}|}=\frac{1/3}{1}.$$

$$\overrightarrow{p_n}=p_{nx}\vec{i}+p_{ny}\vec{j}+p_{nz}\vec{k}$$

$$p_{nx}=\overrightarrow{p_n}\cdot\vec{i}=\overrightarrow{p_x}\cdot\vec{n}=(7\vec{i}-2\vec{k})\cdot\left(\frac{2}{3}\vec{i}-\frac{2}{3}\vec{j}+\frac{1}{3}\vec{k}\right)=7\frac{2}{3}-2\frac{1}{3}=4$$

$$p_{ny}=\overrightarrow{p_n}\cdot\vec{j}=\overrightarrow{p_y}\cdot\vec{n}=(5\vec{j})\cdot\left(\frac{2}{3}\vec{i}-\frac{2}{3}\vec{j}+\frac{1}{3}\vec{k}\right)=5\left(-\frac{2}{3}\right)=-10/3$$

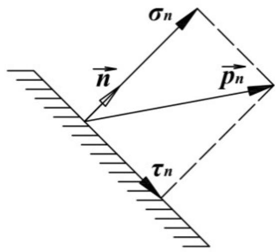
$$p_{nz}=\overrightarrow{p_n}\cdot\vec{k}=\overrightarrow{p_z}\cdot\vec{n}=(-2\vec{i}+4\vec{k})\cdot\left(\frac{2}{3}\vec{i}-\frac{2}{3}\vec{j}+\frac{1}{3}\vec{k}\right)=-2\frac{2}{3}+4\frac{1}{3}=0$$



$$\boxed{\overrightarrow{p_n}=4\vec{i}-\frac{10}{3}\vec{j}}$$ - Vektor totalnog napona

$$|\overrightarrow{p_n}|=\sqrt{p_{nx}^2+p_{ny}^2+p_{nz}^2}=\sqrt{4^2+\left(-\frac{10}{3}\right)^2}=5.207\ Pa$$ - Intenzitet totalnog napona

$$\left.\begin{matrix} \cos\alpha_{\overrightarrow{p_n}}=\frac{p_{nx}}{|\overrightarrow{p_n}|}=\frac{4}{5.207}=0.768 \\ \cos\beta_{\overrightarrow{p_n}}=\frac{p_{ny}}{|\overrightarrow{p_n}|}=\frac{-10/3}{5.207}=-0.640 \\ \cos\gamma_{\overrightarrow{p_n}}=\frac{p_{nz}}{|\overrightarrow{p_n}|}=\frac{0}{5.207}=0 \end{matrix}\right\}$$ - Pravac totalnog napona $\overrightarrow{p_n}$



c. Normalni napon i totalni smičući napon

$$\sigma_n=\overrightarrow{p_n}\cdot\vec{n}=\left(4\vec{i}-\frac{10}{3}\vec{j}\right)\cdot\left(\frac{2}{3}\vec{i}-\frac{2}{3}\vec{j}+\frac{1}{3}\vec{k}\right)=4\frac{2}{3}-\frac{10}{3}\left(-\frac{2}{3}\right)=4.889\,Pa$$ - Normalni napon

$$\tau_n=\sqrt{|\overrightarrow{p_n}|^2-\sigma_n^2}=\sqrt{5.2017^2-4.889^2}=1.792\ Pa$$ - Totalni smičući napon

3. Stanje napona u tački opterećenog tijela zadato je tenzorom napona $[S]$.

$$[S]=\begin{bmatrix}0 & 0 & 0\\ 0 & -30 & -20\\ 0 & -20 & 40\end{bmatrix}\ (kPa)$$

a. Pokazati da je stanje napona ravno;
b. Prikazati okolinu tačke;
c. Sračunati glavne napone i pravce glavnih napona i prikazati okolinu tačke;
d. Sračunati maksimalni smičući napon i odgovarjući normalni napon. Prikazati okolinu tačke;
e. Odrediti komponentalne napone za ravni čije normale leže u ravni *yz* i sa osom *y* grede uglove $\varphi_1=30^0$ odnosno $\varphi_2=-45^0$;
f. Nacrtati *Mohr*-ov krug napona i provjeriti rješenja iz prethodnih tačaka zadatka.

R J E Š E NJ E

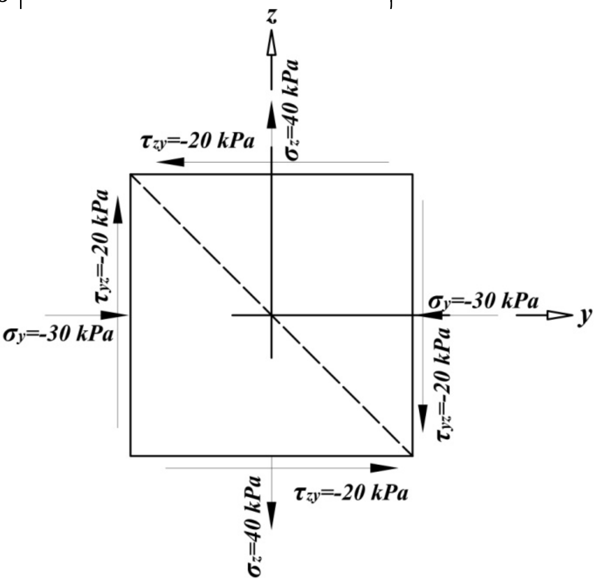
a. Ravno stanje napona: $\det[S]=0$, $Makar\,1\,minor \neq 0$

$$\det[S]=\begin{vmatrix}0 & 0 & 0\\ 0 & -30 & -20\\ 0 & -20 & 40\end{vmatrix}=0$$

$$\det[S_1]=\begin{vmatrix}-30 & -20\\ -20 & 40\end{vmatrix}=-1200-400=-1600\neq 0$$

$\rightarrow$ Stanje napona je ravno

b. Okolina tačke



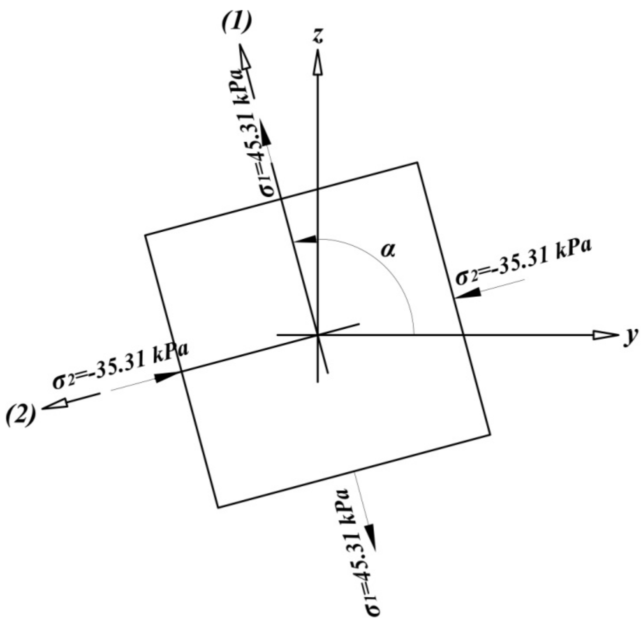
c. Glavni normalni naponi

$$\sigma_{1/2}=\frac{1}{2}\left(\sigma_y+\sigma_z\right)\pm\frac{1}{2}\sqrt{\left(\sigma_y-\sigma_z\right)^2+4\tau_{yz}^{\;2}}=$$
$$=\frac{1}{2}(-30+40)\pm\frac{1}{2}\sqrt{(-30-40)^2+4(-20)^2}=$$
$$=5\pm 40.31$$

$$\left.\begin{aligned}\sigma_1&=45.31\,kPa\\ \sigma_2&=-35.31\,kPa\end{aligned}\right\}$$ - Glavni normalni naponi

Kontrola $\sigma_1+\sigma_2=\sigma_y+\sigma_z$

$$tg2\alpha=\frac{2\tau_{yz}}{\sigma_y-\sigma_z}=\frac{2\cdot(-20)}{-30-40}=\frac{4}{7}$$



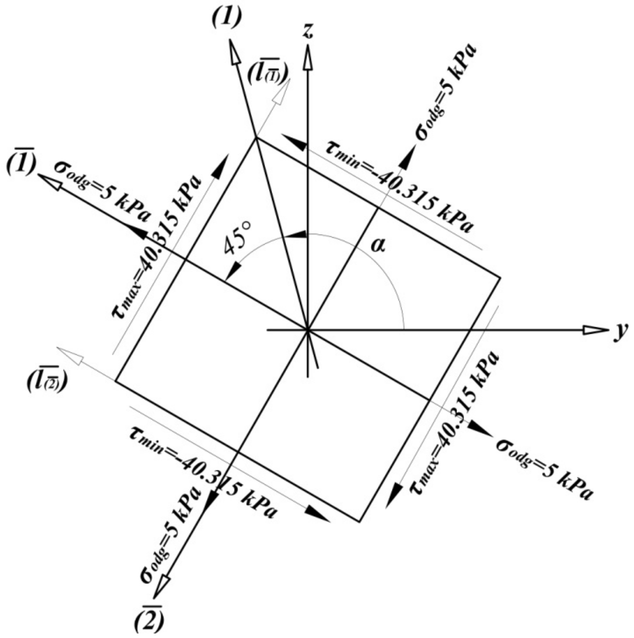
$(\sigma_y-\sigma_z)=-70<0 \quad\rightarrow\quad \alpha=\frac{1}{2}arctg\left(\frac{4}{7}\right)+90°=104.87^0$ - Ugao koji definiše glavni pravac *(1)*

d. Maksimalni smičući napon i odgovarjući normalni napon

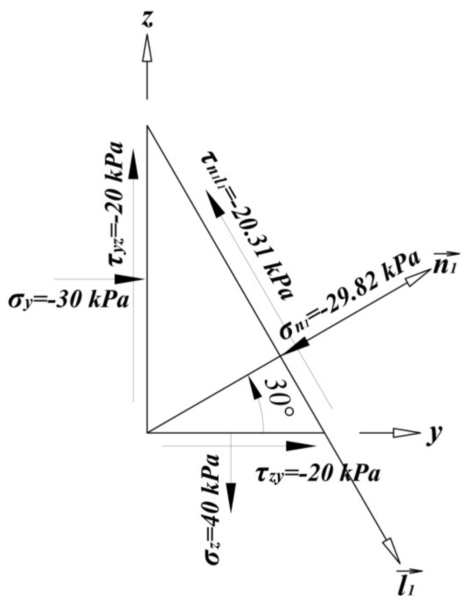
$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_2) = \frac{1}{2}(45.31 - (-35.31)) = 40.31\ kPa$ - Maksimalni smičući napon

$\sigma_s = \sigma_{odg} = \frac{1}{2}(\sigma_1 + \sigma_2) = \frac{1}{2}(45.31 + (-35.31)) = 5\ kPa$ - Odgovarjući normalni napon

$\overline{\alpha} = \alpha + 45^0$ - Ugao koji definiše normalu za ravan u kojoj djeluje maksimalni smičući napon



e. Komponentalni naponi za ravni φ_1=30^0 odnosno φ_2=-45^0



$\overrightarrow{n_1} \leftrightarrow \varphi_1 = 30^0$

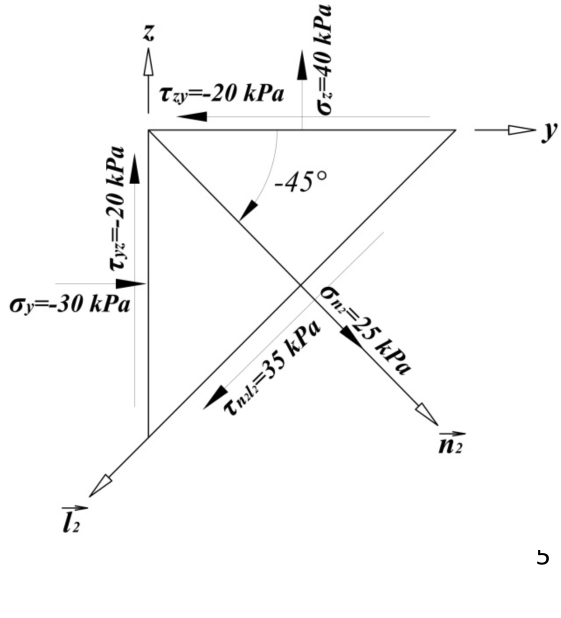
$\sigma_{n_1} = \frac{1}{2}(\sigma_y + \sigma_z) + \frac{1}{2}(\sigma_y - \sigma_z)\cos 2\varphi_1 + \tau_{yz}\sin 2\varphi_1 =$

$= \frac{1}{2}(-30+40) + \frac{1}{2}(-30-40)\cos(2\cdot 30^0) + (-20)\sin(2\cdot 30^0) =$

$= -29.82\ kPa$

$\tau_{n_1 l_1} = \frac{1}{2}(\sigma_y - \sigma_z)\sin 2\varphi_1 - \tau_{yz}\cos 2\varphi_1 =$

$= \frac{1}{2}(-30-40)\sin(2\cdot 30^0) - (-20)\cos(2\cdot 30^0) = -20.31\ kPa$



$\overrightarrow{n_2} \leftrightarrow \varphi_2 = -45^0$

$\sigma_{n_2} = \frac{1}{2}(\sigma_y + \sigma_z) + \frac{1}{2}(\sigma_y - \sigma_z)\cos 2\varphi_2 + \tau_{yz}\sin 2\varphi_2 =$

$= \frac{1}{2}(-30+40) + \frac{1}{2}(-30-40)\cos(2\cdot(-45^0)) + (-20)\sin(2\cdot(-45^0))$

$= 25\ kPa$

$\tau_{n_2 l_2} = \frac{1}{2}(\sigma_y - \sigma_z)\sin 2\varphi_2 - \tau_{yz}\cos 2\varphi_2 =$

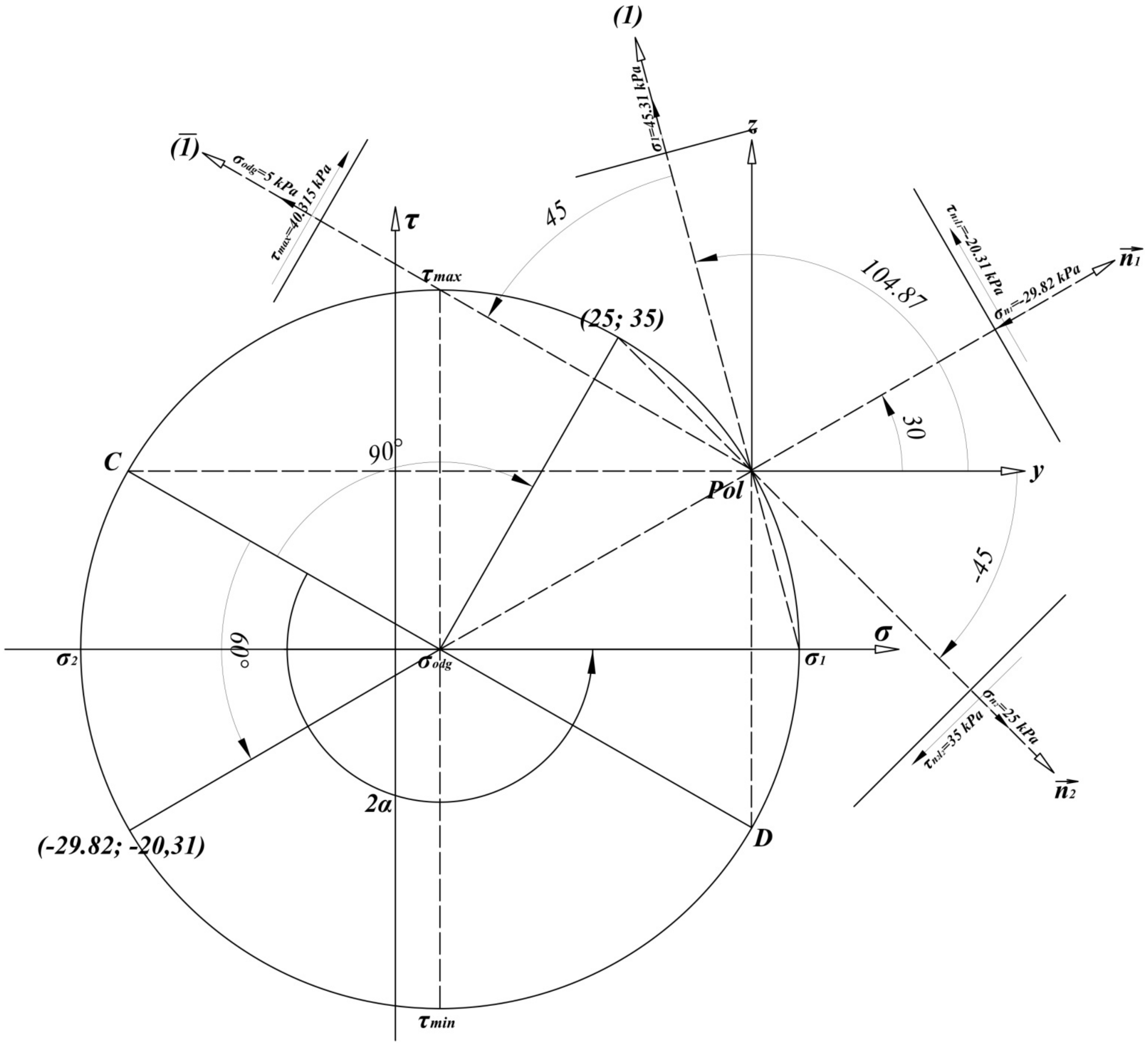
$= \frac{1}{2}(-30-40)\sin(2\cdot(-45^0)) - (-20)\cos(2\cdot(-45^0)) = 35\ kPa$

f. *Mohr*-ov krug napona

$C\left[\sigma_y;-\tau_{yz}\right]=\left[-30;20\right]\ kPa$

$D\left[\sigma_z;\tau_{yz}\right]=\left[40;-20\right]\ kPa$

Usvaja se razmjer 10 *kPa* = 1 *cm*



VJEŽBA BR. 6

ANALIZA NAPONA

1. Stanje napona u tački opterećenog tijela zadato je tenzorom napona [S]. Odrediti:

$$[S]=\begin{bmatrix}-20 & 0 & -6\\ 0 & 6 & 0\\ -6 & 0 & 6\end{bmatrix}\ (MPa)$$

   a. Vektor totalnog napona za ravan sa normalom $\vec{n}=\left\{\frac{1}{3};\frac{2}{3};n_z<0\right\}$;
   b. Normalni i smičući napon u toj ravni;
   c. Glavne napone i glavne pravce;
   d. Maksimalni smičući napon i odgovarajući normalni napon;
   e. Sferni i devijatorski dio tenzora napona.

R J E Š E NJ E

a. Vektor totalnog napona za ravan sa normalom $\vec{n}$

$$|\vec{n}|=\sqrt{\left(\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2+(n_z)^2}=1\Big|^2 \;\rightarrow\; \left(\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2+(n_z)^2=1 \;\rightarrow\; n_z{}^2=\frac{4}{9} \;\rightarrow\; n_z=\pm\frac{2}{3} \qquad \boxed{n_z=-\frac{2}{3}}$$

$$\vec{n}=\left\{\frac{1}{3};\frac{2}{3};-\frac{2}{3}\right\}=\frac{1}{3}\vec{i}+\frac{2}{3}\vec{j}-\frac{2}{3}\vec{k}$$

$$[S]=\begin{bmatrix}-20 & 0 & -6\\ 0 & 6 & 0\\ -6 & 0 & 6\end{bmatrix}\ (MPa) \;\rightarrow\; \begin{aligned}\overrightarrow{p_x}&=-20\vec{i}-6\vec{k}\\ \overrightarrow{p_y}&=6\vec{j}\\ \overrightarrow{p_z}&=-6\vec{i}+6\vec{k}\end{aligned}$$

$$\overrightarrow{p_n}=p_{nx}\vec{i}+p_{ny}\vec{j}+p_{nz}\vec{k}$$

$$p_{nx}=\overrightarrow{p_n}\cdot\vec{i}=\overrightarrow{p_x}\cdot\vec{n}=(-20\vec{i}-6\vec{k})\cdot\left(\frac{1}{3}\vec{i}+\frac{2}{3}\vec{j}-\frac{2}{3}\vec{k}\right)=-20\frac{1}{3}-6\left(-\frac{2}{3}\right)=-\frac{8}{3}$$

$$p_{ny}=\overrightarrow{p_n}\cdot\vec{j}=\overrightarrow{p_y}\cdot\vec{n}=(6\vec{j})\cdot\left(\frac{1}{3}\vec{i}+\frac{2}{3}\vec{j}-\frac{2}{3}\vec{k}\right)=6\frac{2}{3}=4$$

$$p_{nz}=\overrightarrow{p_n}\cdot\vec{k}=\overrightarrow{p_z}\cdot\vec{n}=(-6\vec{i}+6\vec{k})\cdot\left(\frac{1}{3}\vec{i}+\frac{2}{3}\vec{j}-\frac{2}{3}\vec{k}\right)=-6\frac{1}{3}+6\left(-\frac{2}{3}\right)=-6$$

$$\overrightarrow{p_n}=-\frac{8}{3}\vec{i}+4\vec{j}-6\vec{k}$$

b. Normalni i smičući napon

$$|\overrightarrow{p_n}|=\sqrt{\left(\frac{8}{3}\right)^2+4^2+(-6)^2}=7.688\ MPa$$

$$\sigma_n=\overrightarrow{p_n}\cdot\vec{n}=\left(-\frac{8}{3}\vec{i}+4\vec{j}-6\vec{k}\right)\cdot\left(\frac{1}{3}\vec{i}+\frac{2}{3}\vec{j}-\frac{2}{3}\vec{k}\right)=-\frac{8}{3}\frac{1}{3}+4\frac{2}{3}-6\left(-\frac{2}{3}\right)=5.77\,MPa$$

$$\tau_n=\sqrt{|\overrightarrow{p_n}|^2-\sigma_n{}^2}=\sqrt{7.688^2-5.77^2}=5.08\ MPa$$

c. Glavni naponi i pravci glavnih napona **(I način)**

Intenzitet glavnih normalnih napona

$$\boldsymbol{\sigma_i^3 - I_\sigma \sigma_i^2 + II_\sigma \sigma_i - III_\sigma = 0}$$

$$I_\sigma = \sigma_x + \sigma_y + \sigma_z = -20 + 6 + 6 = -8$$

$$II_\sigma = \begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{zy} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xz} \\ \tau_{zx} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{vmatrix} =$$

$$= \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 = (-20)\cdot 6 + 6\cdot 6 + 6\cdot(-20) - (-6)^2 = -240$$

$$III_\sigma = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{vmatrix} = \begin{vmatrix} -20 & 0 & -6 \\ 0 & 6 & 0 \\ -6 & 0 & 6 \end{vmatrix} = -20\cdot(6\cdot 6) - 6\cdot(36) = -936$$

$\boldsymbol{\sigma_i^3} + 8\boldsymbol{\sigma_i^2} - 240\boldsymbol{\sigma_i} + 936 = 0$ $\quad \boxed{\boldsymbol{\sigma_\alpha} = 6\ MPa}$ jedno od 3 rješenja kubne jednačine

$$\left(\boldsymbol{\sigma_i^3} + 8\boldsymbol{\sigma_i^2} - 240\boldsymbol{\sigma_i} + 936\right) : \left(\boldsymbol{\sigma_i} - 6\right) = \boldsymbol{\sigma_i^2} + 14\boldsymbol{\sigma_i} - 156$$

$$\underline{\boldsymbol{\sigma_i^3} - 6\boldsymbol{\sigma_i^2}}$$

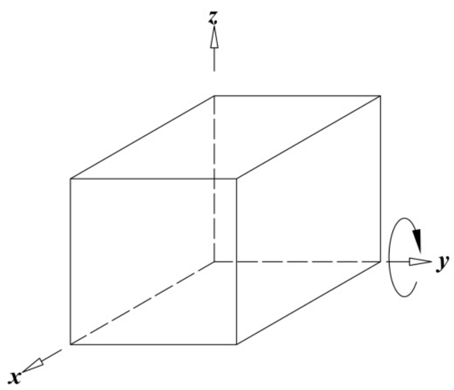
$$14\boldsymbol{\sigma_i^2} - 240\boldsymbol{\sigma_i}$$

$$\underline{14\boldsymbol{\sigma_i^2} - 84\boldsymbol{\sigma_i}}$$

$$-156\boldsymbol{\sigma_i} + 936$$

$$\underline{-156\boldsymbol{\sigma_i} + 936}$$

$$0.0$$



$$\boldsymbol{\sigma_i^2} + 14\boldsymbol{\sigma_i} - 156 = 0$$

$$\boldsymbol{\sigma_{b/c}} = \frac{-14 \pm \sqrt{196 + 4\cdot 156}}{2} - 7 \pm 14.32$$

$$\boldsymbol{\sigma_b} = 7.32\ MPa$$

$$\boldsymbol{\sigma_c} = -21.32\ Pa$$

$$\boldsymbol{\sigma_3 \le \sigma_2 \le \sigma_1} \quad \rightarrow \quad \boxed{\begin{array}{l} \boldsymbol{\sigma_1 = \sigma_b = 7.32\ MPa} \\ \boldsymbol{\sigma_2 = \sigma_a = 6\ MPa} \\ \boldsymbol{\sigma_3 = \sigma_c = -21.32\ MPa} \end{array}}$$

Pravci glavnih normalnih napona $\quad \vec{n_i} = \cos\alpha_i \vec{i} + \cos\beta_i \vec{j} + \cos\gamma_i \vec{k} \quad i = 1,2,3$

$$\cos\alpha_i = \pm\frac{A_i}{D_i};\ \cos\beta_i = \pm\frac{B_i}{D_i};\ \cos\gamma_i = \pm\frac{C_i}{D_i}$$

$$A_i = \begin{vmatrix} (\sigma_y - \sigma_i) & \tau_{yz} \\ \tau_{zy} & (\sigma_z - \sigma_i) \end{vmatrix}; \quad B_i = -\begin{vmatrix} \tau_{yx} & \tau_{yz} \\ \tau_{zx} & (\sigma_z - \sigma_i) \end{vmatrix}; \quad C_i = \begin{vmatrix} \tau_{yx} & (\sigma_y - \sigma_i) \\ \tau_{zx} & \tau_{zy} \end{vmatrix}; \quad D_i = \sqrt{A_i^2 + B_i^2 + C_i^2}$$

Pravac (1)

$$A_1 = \begin{vmatrix} (\sigma_y - \sigma_1) & \tau_{yz} \\ \tau_{zy} & (\sigma_z - \sigma_1) \end{vmatrix} = \begin{vmatrix} (6-7.32) & 0 \\ 0 & (6-7.32) \end{vmatrix} = 1.7424$$

$$B_1 = -\begin{vmatrix} \tau_{yx} & \tau_{yz} \\ \tau_{zx} & (\sigma_z - \sigma_1) \end{vmatrix} = -\begin{vmatrix} 0 & 0 \\ -6 & (6-7.32) \end{vmatrix} = 0$$

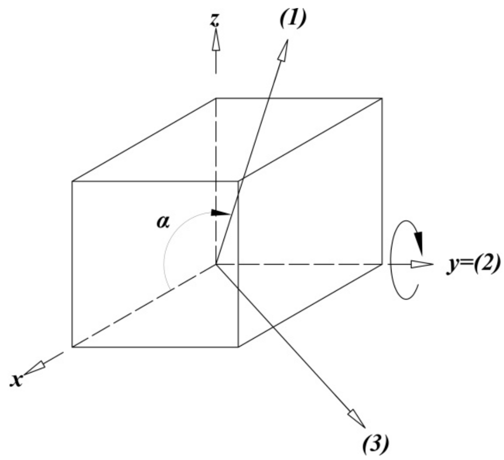
$$C_1 = \begin{vmatrix} \tau_{yx} & (\sigma_y - \sigma_1) \\ \tau_{zx} & \tau_{zy} \end{vmatrix} = \begin{vmatrix} 0 & (6-7.32) \\ -6 & 0 \end{vmatrix} = -7.92$$

$$D_1 = \sqrt{A_1^2 + B_1^2 + C_1^2} = \sqrt{1.7424^2 + 0^2 + (-7.92)^2} = 8.11$$

$$\left.\begin{aligned} \cos\alpha_1 &= \pm\frac{A_1}{D_1} = \pm\frac{1.7421}{8.11} = \pm 0.215 \\ \cos\beta_1 &= \pm\frac{B_1}{D_1} = \pm\frac{0}{8.11} = 0 \\ \cos\gamma_1 &= \pm\frac{C_1}{D_1} = \pm\frac{-7.92}{8.11} = \mp 0.977 \end{aligned}\right\} \rightarrow \boxed{\vec{n_1} = \pm 0.215\vec{i} \mp 0.977\vec{k}}$$

Pravac (2)

$\boldsymbol{\sigma_2 = \sigma_y = 6\ MPa}$ y osa je glavna osa $\rightarrow$ $\boxed{\vec{n_2} = \pm\vec{j}}$



Pravac (3)

$$A_3 = \begin{vmatrix} (\sigma_y - \sigma_3) & \tau_{yz} \\ \tau_{zy} & (\sigma_z - \sigma_3) \end{vmatrix} = \begin{vmatrix} (6+21.32) & 0 \\ 0 & (6+21.32) \end{vmatrix} = 746.38$$

$$B_3 = -\begin{vmatrix} \tau_{yx} & \tau_{yz} \\ \tau_{zx} & (\sigma_z - \sigma_3) \end{vmatrix} = -\begin{vmatrix} 0 & 0 \\ -6 & (6+21.32) \end{vmatrix} = 0$$

$$C_3 = \begin{vmatrix} \tau_{yx} & (\sigma_y - \sigma_3) \\ \tau_{zx} & \tau_{zy} \end{vmatrix} = \begin{vmatrix} 0 & (6+21.32) \\ -6 & 0 \end{vmatrix} = 163.92$$

$$D_3 = \sqrt{A_3^2 + B_3^2 + C_3^2} = \sqrt{746.38^2 + 0^2 + 163.92^2} = 764.17$$

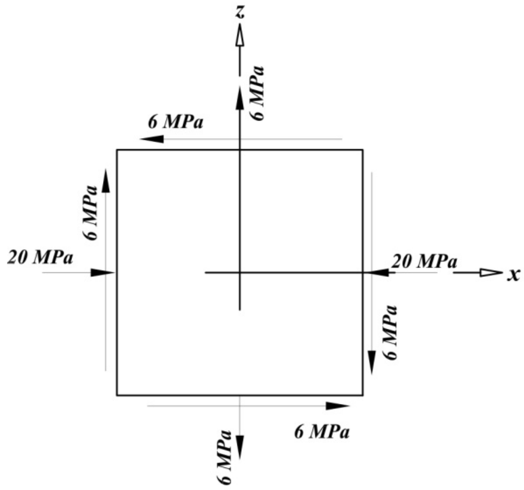
$$\left.\begin{aligned} \cos\alpha_3 &= \pm\frac{A_3}{D_3} = \pm\frac{746.38}{764.17} = \pm 0.977 \\ \cos\beta_3 &= \pm\frac{B_3}{D_3} = \pm\frac{0}{764.17} = 0 \\ \cos\gamma_3 &= \pm\frac{C_3}{D_3} = \pm\frac{163.92}{764.17} = \pm 0.215 \end{aligned}\right\} \rightarrow \boxed{\vec{n_3} = \pm 0.977\vec{i} \pm 0.215\vec{k}}$$

c. Glavni naponi i pravci glavnih napona **(II način)**

$$[S] = \begin{bmatrix} -20 & 0 & -6 \\ 0 & 6 & 0 \\ -6 & 0 & 6 \end{bmatrix}$$ $\rightarrow$ $\boldsymbol{\sigma_y = 6\ MPa}$ je jedan od tri glavna napona, a y osa je jedna od tri glavne osa

Preostale dvije glavne ose, ćemo dobiti rotacijom elementarnog kvadra (beskonačno mala okolina tačke) oko ose y $\longrightarrow$ Možemo iskoristiti formule za proračun glavnih normalnih napona i glavnih osa za ravno stanje napona.

$$[S_{xz}] = \begin{bmatrix} \sigma_x & \tau_{xz} \\ \tau_{zx} & \sigma_z \end{bmatrix} = \begin{bmatrix} -20 & -6 \\ -6 & 6 \end{bmatrix}$$



$$\sigma_{a/b} = \frac{1}{2}(\sigma_x + \sigma_z) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_z)^2 + 4\tau_{xz}^{\ 2}} =$$
$$= \frac{1}{2}(-20+6) \pm \frac{1}{2}\sqrt{(-20-6)^2 + 4(-6)^2} =$$
$$= -7 \pm 14.31$$

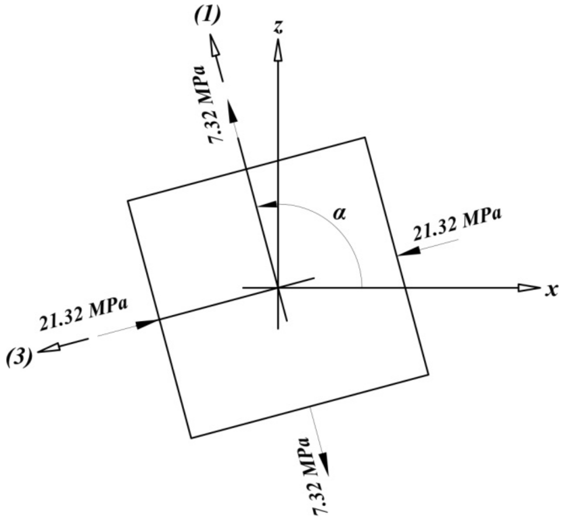
$$\left.\begin{aligned} \sigma_a &= 7.32\ MPa \\ \sigma_b &= -21.32\ MPa \end{aligned}\right\}$$ - Preostala dva glavna normalna naponi

Kontrola $\sigma_a + \sigma_b = \sigma_x + \sigma_z$

$$tg2\alpha = \frac{2\tau_{xz}}{\sigma_x - \sigma_z} = \frac{2\cdot(-6)}{-20-6} = 0.4615$$

$(\sigma_x - \sigma_z) < 0 \quad \rightarrow \quad \alpha = \frac{1}{2}arctg(0.4615) + 90° = 102.38^0$ - Ugao koji definiše glavni pravac *(a)*

Konačno
$$\boxed{\begin{aligned} \boldsymbol{\sigma_1 = \sigma_a = 7.32\ MPa} \\ \boldsymbol{\sigma_2 = \sigma_c = 6\ MPa} \\ \boldsymbol{\sigma_3 = \sigma_b = -21.32\ MPa} \end{aligned}}$$



Kontrola:

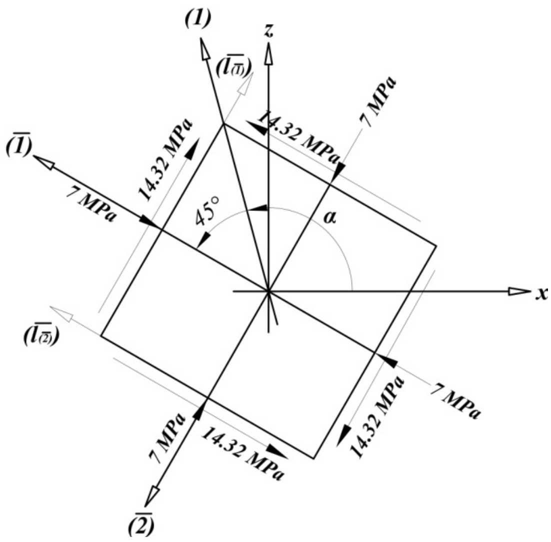
Pravac (1) $\begin{pmatrix} \cos\alpha_1 = \cos 102.38 = -0.215 \\ \cos\beta_1 = \cos 90 = 0 \\ \cos\gamma_1 = \cos(102.38 - 90) = 0.977 \end{pmatrix}$

Pravac (2) $\begin{pmatrix} \cos\alpha_2 = \cos 90 = 0 \\ \cos\beta_2 = \cos 0 = 1 \\ \cos\gamma_2 = \cos 90 = 0 \end{pmatrix}$

Pravac (3) $\begin{pmatrix} \cos\alpha_3 = \cos(102.38 + 90) = -0.977 \\ \cos\beta_3 = \cos 90 = 0 \\ \cos\gamma_3 = \cos 102.38 = -0.215 \end{pmatrix}$

Očigledno je da je II način značajno jednostavniji za primjenu. Naravno, ograničenje je da mora biti poznat jedan od tri glavna pravca i jedan od tri glavna normalna napona.

d. Maksimalni smičući napon i odgovarajući normalni napon



$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3) = \frac{1}{2}(7.32 - (-21.32)) = 14.32\ MPa$$

$$\sigma_s = \sigma_{odg} = \frac{1}{2}(\sigma_1 + \sigma_3) = \frac{1}{2}(7.32 + (-21.32)) = -7\ MPa$$

e. Sferni i devijatorski dio tenzora napona

$$\sigma_s = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) = \frac{1}{3}(-20 + 6 + 6) = -\frac{8}{3}\ MPa$$

$$S = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} (\sigma_x - \sigma_s) & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & (\sigma_y - \sigma_s) & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & (\sigma_y - \sigma_s) \end{bmatrix} + \begin{bmatrix} \sigma_s & 0 & 0 \\ 0 & \sigma_s & 0 \\ 0 & 0 & \sigma_s \end{bmatrix} = S'' + S'$$

$$[S'] = \begin{bmatrix} \sigma_s & 0 & 0 \\ 0 & \sigma_s & 0 \\ 0 & 0 & \sigma_s \end{bmatrix} = \begin{bmatrix} -\frac{8}{3} & 0 & 0 \\ 0 & -\frac{8}{3} & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix} (MPa)$$

$$[S''] = \begin{bmatrix} (\sigma_x - \sigma_s) & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & (\sigma_y - \sigma_s) & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & (\sigma_y - \sigma_s) \end{bmatrix} = \begin{bmatrix} -20 + \frac{8}{3} & 0 & -6 \\ 0 & 6 + \frac{8}{3} & 0 \\ -6 & 0 & 6 + \frac{8}{3} \end{bmatrix} = \begin{bmatrix} -17.33 & 0 & -6 \\ 0 & 8.66 & 0 \\ -6 & 0 & 8.66 \end{bmatrix} (MPa)$$

$[S']$ - Sferni dio tenzora napona (hidrostatički pritisak) - utiče na promjenu zapremine

$[S'']$ Devijatorski dio tenzora napona (čisto smicanje) - utiče na promjenu oblika tijela

2. U tački u kojoj vlada ravno stanje napona poznati su sljedeći podaci:

$\sigma_y = -10\,\frac{kN}{cm^2},\ \sigma_z = 30\,\frac{kN}{cm^2}\ i\ \tau_{nl} = 12.32\frac{kN}{cm^2}.$

Odrediti:

a. Tenzor napona i izdvojiti element na koji djeluju;

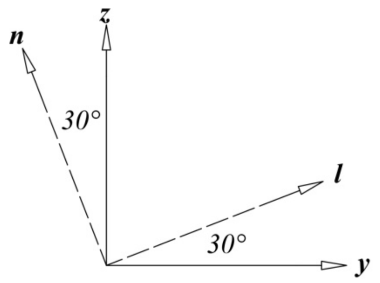
b. Glavne napone i glavne pravce i izdvojiti element na koji djeluju;

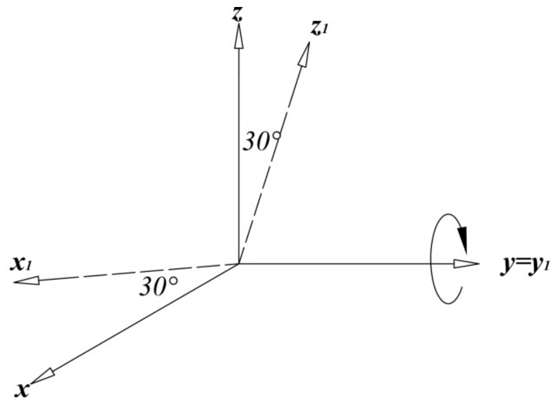
c. Maksimalni smičući napon i odgovarajući normalni napon i izdvojiti element na koji djeluju;

d. Komponentalne napone za ravni sa normalama $\vec{n_1} = \frac{1}{2}\vec{j} - \frac{\sqrt{3}}{2}\vec{k}$ i $\vec{n_2} = \frac{\sqrt{2}}{2}\vec{j} + \frac{\sqrt{2}}{2}\vec{k}$ i izdvojiti element na koji djeluju;

e. Tenzor napona u novom transformisanom koordinatnom sistemu (x_1, y_1, z_1) koji se dobija rotacijom koordinatnog sistema (x_1, y_1, z_1) oko y ose za ugao 30^0;

f. *Mohr*-ov krug napona.





R J E Š E NJ E

a. Tenzor napona

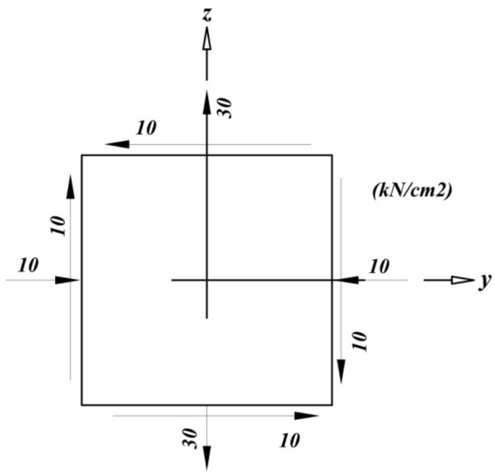
$$[S] = \begin{bmatrix} \sigma_y & \tau_{yz} \\ \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} -10 & \tau_{yz} \\ \tau_{zy} & 30 \end{bmatrix} \quad \tau_{yz} = ?$$

$$\tau_{nl} = \frac{1}{2}(\sigma_y - \sigma_z)\sin 2\varphi - \tau_{yz}\cos 2\varphi$$

$$12.32 = \frac{1}{2}(-10-30)\sin(2\cdot 120^0) - (\tau_{yz})\cos(2\cdot 120^0)$$

$$\tau_{yz} = -10\frac{kN}{cm^2}$$

$$\boxed{[S] = \begin{bmatrix} -10 & -10 \\ -10 & 30 \end{bmatrix} \frac{kN}{cm^2}}$$



b. Glavni naponi i glavni pravci

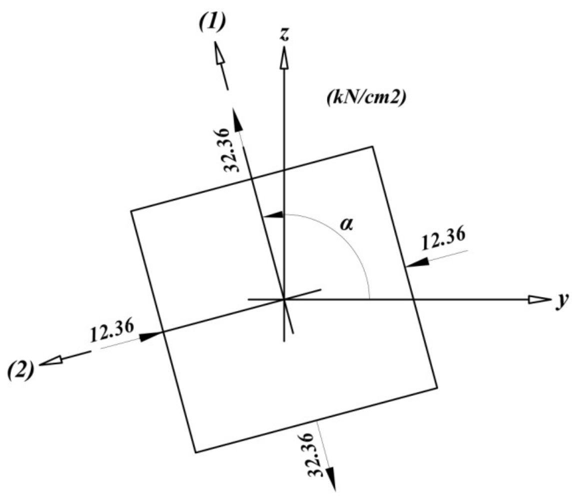
$$\sigma_{1/2} = \frac{1}{2}(\sigma_y + \sigma_z) \pm \frac{1}{2}\sqrt{(\sigma_y - \sigma_z)^2 + 4\tau_{yz}^{\ 2}} =$$

$$= \frac{1}{2}(-10+30) \pm \frac{1}{2}\sqrt{(-10-30)^2 + 4(-10)^2} =$$

$$= 10 \pm 22.36$$

$$\left.\begin{array}{l} \sigma_1 = 32.36\,\frac{kN}{cm^2} \\ \sigma_2 = -12.36\,\frac{kN}{cm^2} \end{array}\right\} \text{ - Glavni normalni naponi}$$

Kontrola $\sigma_1 + \sigma_2 = \sigma_y + \sigma_z$



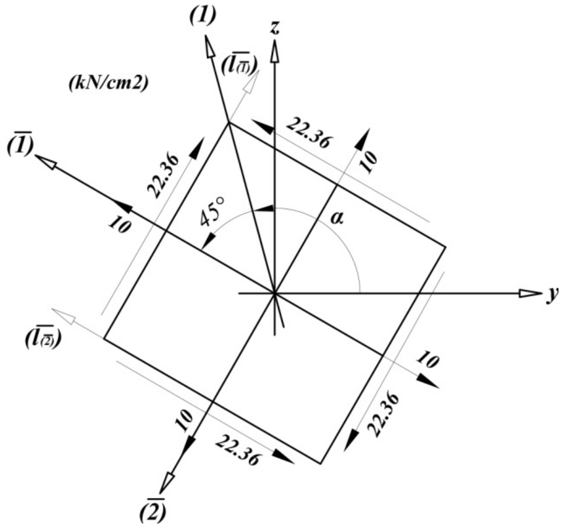
$$tg2\alpha = \frac{2\tau_{yz}}{\sigma_y - \sigma_z} = \frac{2\cdot(-10)}{-10-30} = \frac{1}{2}$$

$(\sigma_y - \sigma_z) = -40 < 0 \quad \rightarrow \quad \alpha = \frac{1}{2}arctg\left(\frac{1}{2}\right) + 90° = 103.28^0$ - Ugao koji definiše glavni pravac *(1)*

c. Maksimalni smičući napon i odgovarajući normalni napon

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_2) = \frac{1}{2}\big(32.36 - (-12.36)\big) = 22.36\ \frac{kN}{cm^2}$$

$$\sigma_s = \sigma_{odg} = \frac{1}{2}(\sigma_1 + \sigma_2) = \frac{1}{2}\big(32.36 + (-12.36)\big) = 10\ \frac{kN}{cm^2}$$



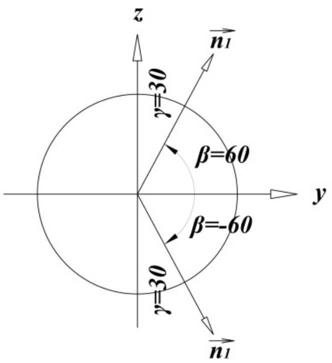
d. Komponentalni naponi za ravni sa normalama $\vec{n_1}$ i $\vec{n_2}$

$$\vec{n_1} = \frac{1}{2}\vec{j} - \frac{\sqrt{3}}{2}\vec{k} = \cos\beta\,\vec{j} + \cos\gamma\,\vec{k}$$

$\beta -$ *ugao koji* $\vec{n_1}$ *gradi sa "y" osom*

$\gamma -$ *ugao koji* $\vec{n_1}$ *gradi sa "z" osom*

$$\left.\begin{aligned}\cos\beta &= \frac{1}{2}\\ \cos\gamma &= -\frac{\sqrt{3}}{2}\end{aligned}\right\} \rightarrow \varphi_1 = -60^0$$



$$\boxed{\vec{n_1} \leftrightarrow \varphi_1 = -60^0}$$

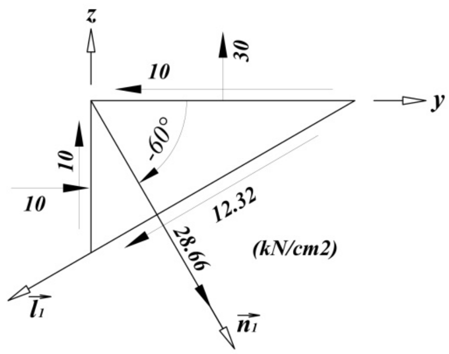
$$\sigma_{n_1} = \frac{1}{2}(\sigma_y + \sigma_z) + \frac{1}{2}(\sigma_y - \sigma_z)\cos 2\varphi_1 + \tau_{yz}\sin 2\varphi_1 =$$

$$= \frac{1}{2}(-10+30) + \frac{1}{2}(-10-30)\cos\big(2\cdot(-60^0)\big) + (-10)\sin\big(2\cdot(-60^0)\big) =$$

$$= 28.66\ \frac{kN}{cm^2}$$

$$\tau_{n_1 l_1} = \frac{1}{2}(\sigma_y - \sigma_z)\sin 2\varphi_1 - \tau_{yz}\cos 2\varphi_1 =$$

$$= \frac{1}{2}(-10-30)\sin\big(2\cdot(-60^0)\big) - (-10)\cos\big(2\cdot(-60^0)\big) = 12.32\ \frac{kN}{cm^2}$$

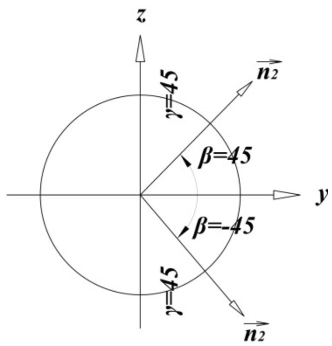


$$\vec{n_2} = \frac{\sqrt{2}}{2}\vec{j} + \frac{\sqrt{2}}{2}\vec{k} = \cos\beta\,\vec{j} + \cos\gamma\,\vec{k}$$

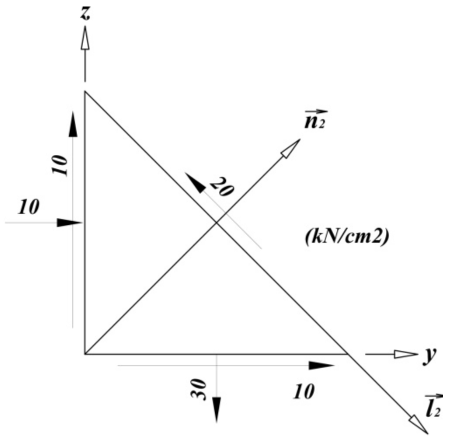
$\beta -$ *ugao koji* $\vec{n_2}$ *gradi sa "y" osom*

$\gamma -$ *ugao koji* $\vec{n_2}$ *gradi sa "z" osom*

$$\left.\begin{aligned}\cos\beta &= \frac{\sqrt{2}}{2}\\ \cos\gamma &= \frac{\sqrt{2}}{2}\end{aligned}\right\} \rightarrow \varphi_2 = 45^0$$



$\boxed{\overrightarrow{n_2} \;\leftrightarrow\; \varphi_2 = 45^0}$



$$\sigma_{n_2} = \frac{1}{2}\left(\sigma_y + \sigma_z\right) + \frac{1}{2}\left(\sigma_y - \sigma_z\right)\cos 2\varphi_2 + \tau_{yz}\sin 2\varphi_2 =$$

$$= \frac{1}{2}\left(-10+30\right) + \frac{1}{2}\left(-10-30\right)\cos\left(2\cdot 45^0\right) + \left(-10\right)\sin\left(2\cdot 45^0\right) =$$

$$= 0\ \frac{kN}{cm^2}$$

$$\tau_{n_2 l_2} = \frac{1}{2}\left(\sigma_y - \sigma_z\right)\sin 2\varphi_2 - \tau_{yz}\cos 2\varphi_2 =$$

$$= \frac{1}{2}\left(-10-30\right)\sin\left(2\cdot 45^0\right) - \left(-10\right)\cos\left(2\cdot 45^0\right) = -20\ \frac{kN}{cm^2}$$

e. Tenzor napona u novom transformisanom koordinatnom sistemu (x_1, y_1, z_1)

$$[S'] = [A][S][A]^T$$

$$[A] = \begin{bmatrix} \cos\alpha_{x_1} & \cos\beta_{x_1} & \cos\gamma_{x_1} \\ \cos\alpha_{y_1} & \cos\beta_{y_1} & \cos\gamma_{y_1} \\ \cos\alpha_{z_1} & \cos\beta_{z_1} & \cos\gamma_{z_1} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix}$$

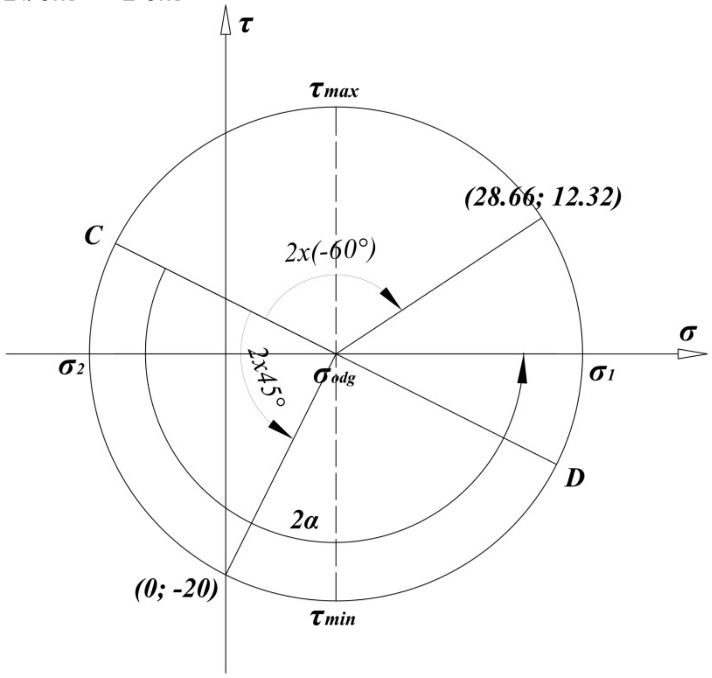
$$[S'] = [A][S][A]^T = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -10 & -10 \\ 0 & -10 & 30 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix} = \left[\qquad\qquad \right]$$

f. *Mohr*-ov krug napona

$$C\left[\sigma_y; -\tau_{yz}\right] = \left[-10;10\right]\ \frac{kN}{cm^2}$$

$$D\left[\sigma_z; \tau_{yz}\right] = \left[30;-10\right]\ \frac{kN}{cm^2}$$

Usvaja se razmjer 10 kN/cm^2 = 1 cm



VJEŽBA BR. 7

ANALIZA DEFORMACIJE

1. Stanje deformacije u tački opterećenog tijela zadato je tenzorom deformacije $[D]$.

   $$[D]=\begin{bmatrix}5 & -1 & -1\\ -1 & 4 & 0\\ -1 & 0 & 4\end{bmatrix}\left(\cdot 10^{-4}\right)$$

   Odrediti:
   a. Dilataciju pravca $\vec{n}$ koji sa koordinatnim osama gradi jednake uglove;
   b. Klizanje između pravca $\vec{n}$ i njemu upravnog pravca $\vec{l}$ koji leži u ravni (x, y);
   c. Tenzor deformacije u novom transformisanom koordinatnom sistemu $\vec{n}, \vec{l}, \vec{m}$, pri čemu $\vec{m}$ gradi oštar ugao sa osom x.

R J E Š E NJ E

a.

$$[D]=\begin{bmatrix}5 & -1 & -1\\ -1 & 4 & 0\\ -1 & 0 & 4\end{bmatrix}\left(\cdot 10^{-4}\right) \rightarrow \begin{matrix}\overrightarrow{d_x}=5\vec{i}-\vec{j}-\vec{k}\\ \overrightarrow{d_y}=-\vec{i}+4\vec{j}\\ \overrightarrow{d_z}=-\vec{i}+4\vec{k}\end{matrix}\quad\left(\cdot 10^{-4}\right)$$

$$\vec{n}=\cos\alpha\vec{i}+\cos\beta\vec{j}+\cos\gamma\vec{k},$$

$$\cos\alpha=\cos\beta=\cos\gamma$$

$$\left|\vec{n}\right|=\sqrt{\cos^2\alpha+\cos^2\alpha+\cos^2\alpha}=1$$

$$\cos\alpha=\pm\frac{1}{\sqrt{3}}$$

$$\vec{n}=\frac{1}{\sqrt{3}}\vec{i}+\frac{1}{\sqrt{3}}\vec{j}+\frac{1}{\sqrt{3}}\vec{k}$$

$$\overrightarrow{d_n}=d_{nx}\vec{i}+d_{ny}\vec{j}+d_{nz}\vec{k}$$

$$d_{nx}=\overrightarrow{d_x}\cdot\vec{n}=(5\vec{i}-\vec{j}-\vec{k})\cdot\left(\frac{1}{\sqrt{3}}\vec{i}+\frac{1}{\sqrt{3}}\vec{j}+\frac{1}{\sqrt{3}}\vec{k}\right)=\frac{5}{\sqrt{3}}-\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}}=\frac{3}{\sqrt{3}}=\sqrt{3}\ \left(\cdot 10^{-4}\right)$$

$$d_{ny}=\overrightarrow{d_y}\cdot\vec{n}=(-\vec{i}+4\vec{j})\cdot\left(\frac{1}{\sqrt{3}}\vec{i}+\frac{1}{\sqrt{3}}\vec{j}+\frac{1}{\sqrt{3}}\vec{k}\right)=-\frac{1}{\sqrt{3}}+\frac{4}{\sqrt{3}}=\frac{3}{\sqrt{3}}=\sqrt{3}\ \left(\cdot 10^{-4}\right)$$

$$d_{nz}=\overrightarrow{d_z}\cdot\vec{n}=(-\vec{i}+4\vec{k})\cdot\left(\frac{1}{\sqrt{3}}\vec{i}+\frac{1}{\sqrt{3}}\vec{j}+\frac{1}{\sqrt{3}}\vec{k}\right)=-\frac{1}{\sqrt{3}}+\frac{4}{\sqrt{3}}=\frac{3}{\sqrt{3}}=\sqrt{3}\ \left(\cdot 10^{-4}\right)$$

$\overrightarrow{d_n}=\sqrt{3}\vec{i}+\sqrt{3}\vec{j}+\sqrt{3}\vec{k}\ \left(\cdot 10^{-4}\right)$ - vektor deformacije za ravan sa normalom $\vec{n}$

$$\varepsilon_n=\overrightarrow{d_n}\cdot\vec{n}=\left(\sqrt{3}\vec{i}+\sqrt{3}\vec{j}+\sqrt{3}\vec{k}\right)\cdot\left(\frac{1}{\sqrt{3}}\vec{i}+\frac{1}{\sqrt{3}}\vec{j}+\frac{1}{\sqrt{3}}\vec{k}\right)=3\cdot 10^{-4}$$

b.

1. $\vec{l}=l_x\vec{i}+l_y\vec{j}+l_z\vec{k},\quad (l_z=0)$ - vektor $\vec{l}$ leži u ravni (x, y);
2. $\vec{n}\cdot\vec{l}=0$ - uslov ortogonalnosti dva pravca;

$$\vec{n}\cdot\vec{l}=\left(\frac{1}{\sqrt{3}}\vec{i}+\frac{1}{\sqrt{3}}\vec{j}+\frac{1}{\sqrt{3}}\vec{k}\right)\cdot\left(l_x\vec{i}+l_y\vec{j}\right)=\frac{l_x}{\sqrt{3}}+\frac{l_y}{\sqrt{3}}=0\quad\rightarrow\quad l_x=-l_y$$

3. $\left|\vec{l}\right| = \sqrt{l_x^2 + l_y^2 + l_z^2} = 1 \rightarrow l_x^2 + l_x^2 = 1 \rightarrow l_x = \pm\frac{1}{\sqrt{2}}$

$\vec{l} = \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j} \quad ili \quad \vec{l} = -\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$

$$\frac{1}{2}\gamma_{nl} = \vec{d_n} \cdot \vec{l} = \left(\sqrt{3}\vec{i} + \sqrt{3}\vec{j} + \sqrt{3}\vec{k}\right) \cdot \left(\frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}\right) = 0$$

c. $\vec{m} = m_x\vec{i} + m_y\vec{j} + m_z\vec{k}$

1. $\vec{m} \cdot \vec{l} = 0 \rightarrow \left(m_x\vec{i} + m_y\vec{j} + m_z\vec{k}\right) \cdot \left(\frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}\right) = m_x\frac{1}{\sqrt{2}} - m_y\frac{1}{\sqrt{2}} = 0 \rightarrow m_x = m_y$

2. $\vec{m} \cdot \vec{n} = 0 \rightarrow \left(m_x\vec{i} + m_y\vec{j} + m_z\vec{k}\right) \cdot \left(\frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k}\right) =$

$= m_x\frac{1}{\sqrt{3}} + m_y\frac{1}{\sqrt{3}} + m_z\frac{1}{\sqrt{3}} = 2m_x\frac{1}{\sqrt{3}} + m_z\frac{1}{\sqrt{3}}0 \rightarrow 2m_x + m_z = 0 \rightarrow m_z = -2m_x$

3. $\left|\vec{m}\right| = \sqrt{m_x^2 + m_y^2 + m_z^2} = 1 \rightarrow 2m_x^2 + m_z^2 = 2m_x^2 + \left(-2m_x\right)^2 = 1 \rightarrow m_x = \pm\frac{1}{\sqrt{6}}$

$\vec{m}$ gradi oštar ugao sa osom x $\rightarrow m_x = \frac{1}{\sqrt{6}} \rightarrow m_y = \frac{1}{\sqrt{6}} \quad m_z = -\frac{2}{\sqrt{6}}$

$\vec{m} = \frac{1}{\sqrt{6}}\vec{i} + \frac{1}{\sqrt{6}}\vec{j} - \frac{2}{\sqrt{6}}\vec{k}$

$$[D'] = [A][D][A]^T$$

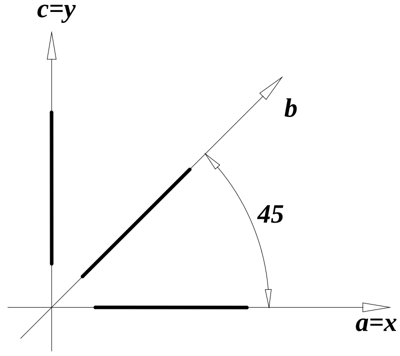
$$[A] = \begin{bmatrix} n_x & n_y & n_z \\ l_x & l_y & l_z \\ m_x & m_y & m_z \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}$$

$$[D'] = [A][D][A]^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 5 & -1 & -1 \\ -1 & 4 & 0 \\ -1 & 0 & 4 \end{bmatrix} (\cdot 10^{-4}) \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5.5 & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & 4.5 \end{bmatrix} (\cdot 10^{-4})$$

2. U tački napregnutog tijela u kojoj vlada ravno stanje deformacije pomoću mjernih traka su izmjerene sljedeće dilatacije: $\varepsilon_a = -5\cdot 10^{-4}$; $\varepsilon_b = 2\cdot 10^{-4}$; $\varepsilon_c = 3\cdot 10^{-4}$.

Odrediti:

a. Tenzor deformacije u toj tački i prikazati deformisanu okolinu tačke;
b. Dilataciju u pravcu koji zaklapa ugao 60^0 sa osom x;
c. Klizanje između tog i njemu upravnog pravca;
d. Glavne dilatacije, pravce glavnih dilatacija i prikazati deformisanu okolinu tačke u sistemu glavnih osa;
e. Maksimalnu smičuću deformaciju (klizanje) i uglove između kojih se događa;
f. *Mohr*-ov krug deformacije.



R J E Š E NJ E

a.

$$\varepsilon_n = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\varphi + \frac{1}{2}\gamma_{xy}\sin 2\varphi$$

$$\varepsilon_b = 2\cdot 10^{-4} \quad (\varphi = 45^0)$$

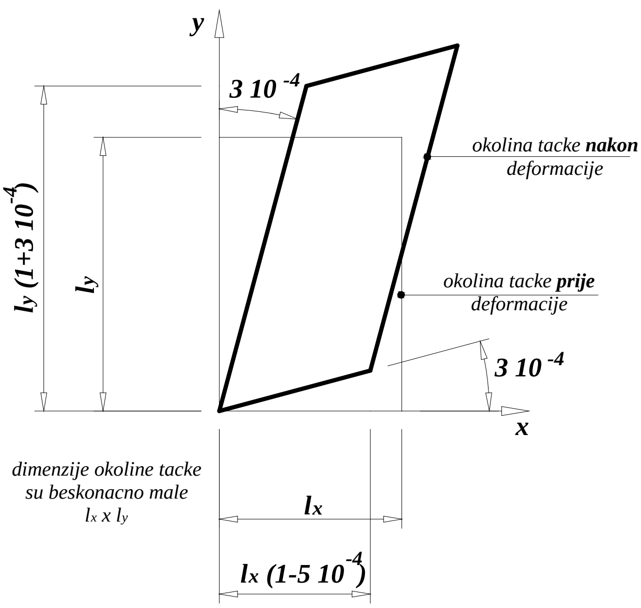
$$\varepsilon_b = \frac{1}{2}(-5+3)10^{-4} + \frac{1}{2}(-5-3)10^{-4}\cos\left(2\cdot 45^0\right) + \frac{1}{2}\gamma_{xy}\cdot 10^{-4}\sin\left(2\cdot 45^0\right) = 2\cdot 10^{-4} \quad \Big| /10^{-4} \quad \rightarrow \quad \frac{1}{2}\gamma_{xy} = 3\cdot 10^{-4}$$

$$[D] = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 3 & 3 \end{bmatrix}\cdot 10^{-4}$$

l_x - elementarna dužina u pravcu x <u>prije</u> deformacije

$\varepsilon_x = \dfrac{\Delta l_x}{l_x}$ - dilatacija u pravcu x

$l_x' = l_x + \Delta l_x = l_x + \varepsilon_x l_x = l_x\left(1+\varepsilon_x\right)$ - elementarna dužina <u>nakon</u> deformacije u pravcu x



b.

$$\vec{n} \leftrightarrow \varphi = 60^0$$

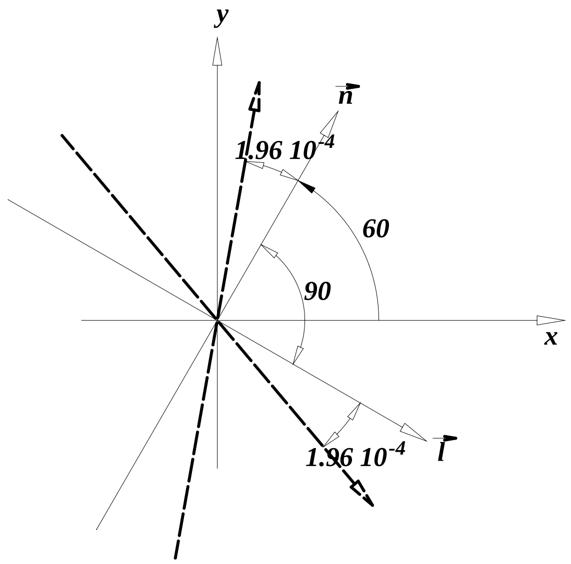
$$\varepsilon_n = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\varphi + \frac{1}{2}\gamma_{xy}\sin 2\varphi =$$

$$= \frac{1}{2}(-5+3)10^{-4} + \frac{1}{2}(-5-3)10^{-4}\cos\left(2\cdot 60^0\right) + 3\cdot 10^{-4}\sin\left(2\cdot 60^0\right) = 3.6\cdot 10^{-4} \ \textit{izduženje}$$

c.

$$\frac{1}{2}\gamma_{nl} = \frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\varphi - \frac{1}{2}\gamma_{xy}\cos 2\varphi =$$

$$= \frac{1}{2}(-5-3)\cdot 10^{-4}\sin\left(2\cdot 60^0\right) - 3\cdot 10^{-4}\cos\left(2\cdot 60^0\right) = -1.96\cdot 10^{-4} \ \ \textit{pravi ugao se povećao}$$



$$\left(\begin{array}{l} \frac{1}{2}\gamma_{\ln}(\varphi = 150^0) = \frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\varphi - \frac{1}{2}\gamma_{xy}\cos 2\varphi = \\ \quad = \frac{1}{2}(-5-3)\cdot 10^{-4}\sin\left(2\cdot 150^0\right) - 3\cdot 10^{-4}\cos\left(2\cdot 150^0\right) = +1.96\cdot 10^{-4} \ \ \textit{pravi ugao se smanjio} \end{array}\right)$$

d.

$$\varepsilon_{1/2} = \frac{1}{2}\left(\varepsilon_x + \varepsilon_y\right) \pm \frac{1}{2}\sqrt{\left(\varepsilon_x - \varepsilon_y\right)^2 + \gamma_{xy}^2} =$$

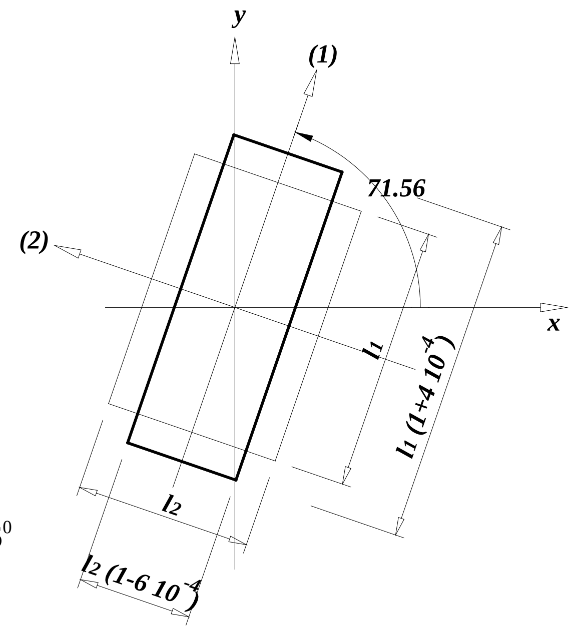
$$= \frac{1}{2}(-5+3)\cdot 10^{-4} \pm \frac{1}{2}\cdot 10^{-4}\sqrt{(-5-3)^2 + 6^2} = (-1 \pm 5)\cdot 10^{-4}$$

$$\varepsilon_1 = 4\cdot 10^{-4}$$

$$\varepsilon_2 = -6\cdot 10^{-4}$$

$$tg 2\alpha = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = \frac{6\cdot 10^{-4}}{(-5-3)\cdot 10^{-4}} = \frac{6}{-8}$$

$$(\varepsilon_x - \varepsilon_y) = (-5-3)\cdot 10^{-4} < 0 \ \rightarrow \ \ \alpha = \frac{1}{2} arctg\left(\frac{6}{-8}\right) + 90° = 71.56^0$$

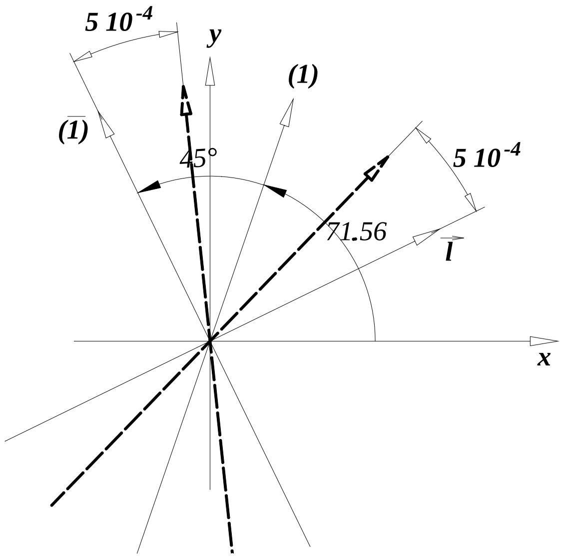


e.

$$\frac{1}{2}\gamma_{max} = \frac{1}{2}(\varepsilon_1 - \varepsilon_2) = \frac{1}{2}(4-(-6))\cdot 10^{-4} = 5\cdot 10^{-4}$$

$$\varepsilon_{odg} = \varepsilon_s = \frac{1}{2}(\varepsilon_1 + \varepsilon_2) = \frac{1}{2}(4+(-6))\cdot 10^{-4} = -1\cdot 10^{-4}$$

$$\overline{\alpha} = \alpha + 45^0$$

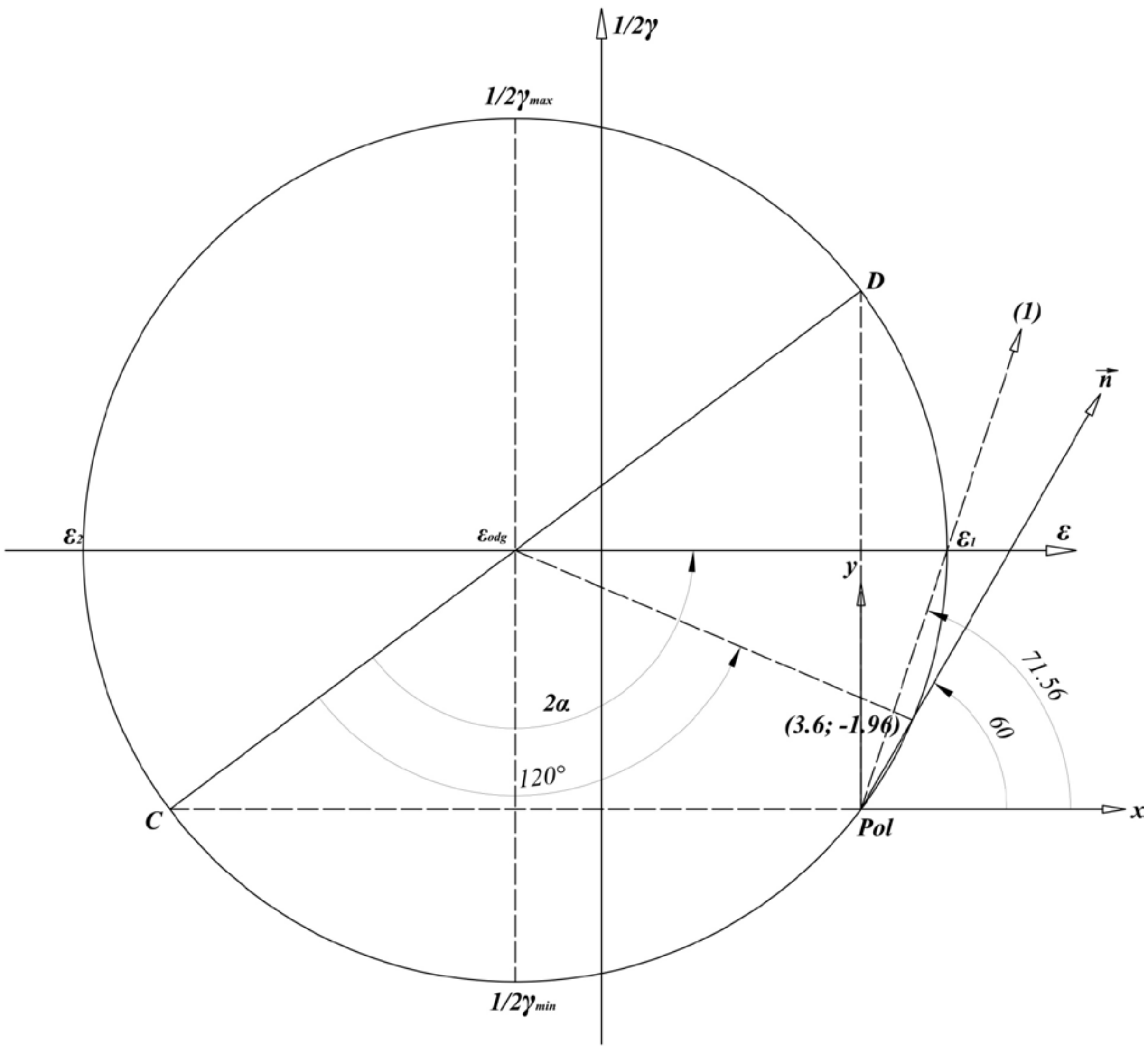


f.

$$C\left[\varepsilon_x;\ -\frac{1}{2}\gamma_{xy}\right] = [-5;\ -3]\cdot 10^{-4}$$

$$D\left[\varepsilon_y;\ \frac{1}{2}\gamma_{xy}\right] = [3;\ 3]\cdot 10^{-4}$$

Usvaja se razmjer $10^{-4} = 1$ *cm*



VJEŽBA BR. 8

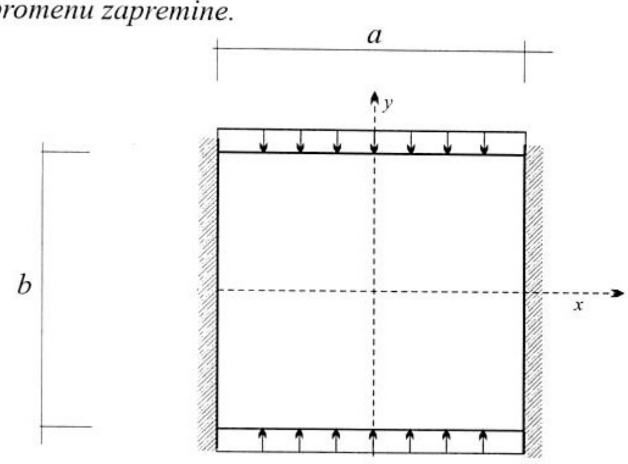
VEZE IZMEĐU NAPONA I DEFORMACIJA

1.

ZADATAK:

Pravougaona ploča a x b = 100 x 100 cm debljine t = 10 cm koja je uklještena između dva idealno kruta i glatka zida opterećena je pritiskom od 10 MPa. Odrediti napone i deformacije u ploči ako se ona zagreje za $\Delta t = 50^{\circ}C$ *kao i specifiču i apsolutnu promenu zapremine.*

E = 210 GPa,
ν = 0.3
α = 1.25 10⁻⁵ C⁻¹



R J E Š E NJ E

Početni (granični) uslovi:

$\boldsymbol{\sigma_z = 0}$	$\boldsymbol{\varepsilon_z \neq 0}$
$\boldsymbol{\sigma_y = -10\ MPa}$	$\boldsymbol{\varepsilon_y \neq 0}$
$\boldsymbol{\sigma_x \neq 0}$	$\boldsymbol{\varepsilon_x = 0}$

$$\tau_{xy} = 0 \qquad \gamma_{xy} = 0$$
$$\tau_{xz} = 0 \quad \rightarrow \quad \gamma_{xz} = 0$$
$$\tau_{yz} = 0 \qquad \gamma_{yz} = 0$$

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \upsilon\left(\sigma_y + \sigma_z\right)\right] + \alpha_t \Delta t = \frac{1}{E}\left[\sigma_x - \upsilon\sigma_y\right] + \alpha_t \Delta t = 0$$

$$\sigma_x = \upsilon\sigma_y - E\alpha_t\Delta t = 0.3\left(-10\right) - 210\cdot 10^3 \cdot 1.25\cdot 10^{-5}\cdot 50 = -134.58\ MPa$$

$$\left[S\right] = \begin{bmatrix} -134.58 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & 0 \end{bmatrix} (MPa)$$ - tenzor napona

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \upsilon\left(\sigma_z + \sigma_x\right)\right] + \alpha_t \Delta t = \frac{1}{210\cdot 10^3}\left[-10 - 0.3\left(0 - 134.58\right)\right] + 1.25\cdot 10^{-5}\cdot 50 = 0.77\cdot 10^{-3}$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \upsilon\left(\sigma_x + \sigma_y\right)\right] + \alpha_t \Delta t = \frac{1}{210\cdot 10^3}\left[0 - 0.3\left(-10 - 134.58\right)\right] + 1.25\cdot 10^{-5}\cdot 50 = 0.83\cdot 10^{-3}$$

$$\left[D\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.77 & 0 \\ 0 & 0 & 0.83 \end{bmatrix}\cdot 10^{-3}$$ - tenzor deformacije

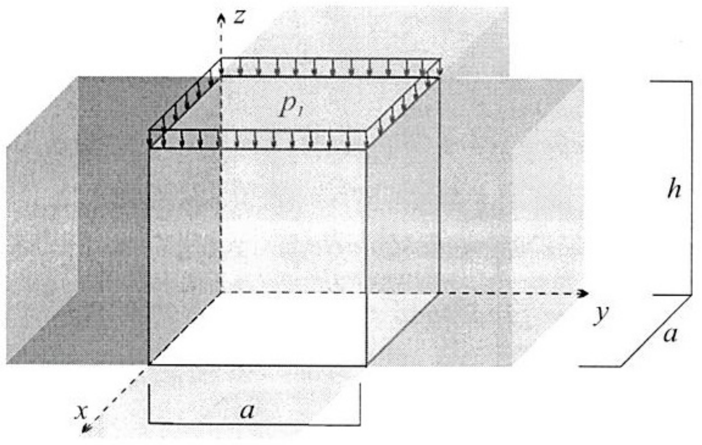
$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = \left(0 + 0.73 + 0.83\right)\cdot 10^{-3} = 1.6\cdot 10^{-3}$ - specifična promjena zapremine

$\Delta V = e\cdot V = 1.6\cdot 10^{-3}\cdot 1\cdot 1\cdot 0.1 = 0.16\cdot 10^{-3}\ m^3$ - apsolutna promjena zapremine

2.

ZADATAK:

Blok materijala na slici sabija se pritiskom p = 50 MPa u bunaru čiji su zidovi kruti i istovremeno se zagreje za Δ T = 50° C. Zanemarujući trenje između materijala i zidova bunara, odrediti:



a/ komponentalne napone i deformacije,
b/ komponentalna pomeranja,
c/ promenu visine bloka,
d/ promenu zapremine bloka,
e/ vrednost kubne dilatacije

Poznato je:
E = 200 GPa, ν = 1/3, α = 1.25 10⁻⁵ C⁻¹
a = 0.1 m, h = 1 m

R J E Š E NJ E

a. komponente napona i deformacije

$\boldsymbol{\sigma_z = -50\ MPa}$	$\boldsymbol{\varepsilon_z \neq 0}$
$\boldsymbol{\sigma_y \neq 0}$	$\boldsymbol{\varepsilon_y = 0}$
$\boldsymbol{\sigma_x \neq 0}$	$\boldsymbol{\varepsilon_x = 0}$

$$\tau_{xy} = 0 \qquad \gamma_{xy} = 0$$
$$\tau_{xz} = 0 \quad \rightarrow \quad \gamma_{xz} = 0$$
$$\tau_{yz} = 0 \qquad \gamma_{yz} = 0$$

$$\left.\begin{aligned} \varepsilon_x &= \frac{1}{E}\left[\sigma_x - \upsilon\left(\sigma_y + \sigma_z\right)\right] + \alpha_t \Delta t = 0 \\ \varepsilon_y &= \frac{1}{E}\left[\sigma_y - \upsilon\left(\sigma_z + \sigma_x\right)\right] + \alpha_t \Delta t = 0 \end{aligned}\right\} \rightarrow \begin{aligned} \sigma_x &= -E\alpha_t \Delta t + \upsilon\left(\sigma_y + \sigma_z\right) \\ \sigma_y &= -E\alpha_t \Delta t + \upsilon\left(\sigma_x + \sigma_z\right) \end{aligned}$$

$$\sigma_y = -E\alpha_t \Delta t + \upsilon\left(-E\alpha_t \Delta t + \upsilon\left(\sigma_y + \sigma_z\right) + \sigma_z\right)$$

$$\sigma_y - \upsilon^2 \sigma_y = -E\alpha_t \Delta t - \upsilon E\alpha_t \Delta t + \upsilon^2 \sigma_z + \upsilon \sigma_z$$

$$\sigma_y = \frac{1}{1-\upsilon^2}\left[-E\alpha_t \Delta t - \upsilon E\alpha_t \Delta t + \upsilon^2 \sigma_z + \upsilon \sigma_z\right] =$$

$$= \frac{1}{1-\frac{1}{9}}\left[-200 \cdot 10^3 \cdot 1.25 \cdot 10^{-5} \cdot 50 - \frac{1}{3} 200 \cdot 10^3 \cdot 1.25 \cdot 10^{-5} \cdot 50 + \frac{1}{9}(-50) + \frac{1}{3}(-50)\right] = -212.5\ MPa$$

$$\sigma_x = -E\alpha_t \Delta t + \upsilon\left(\sigma_y + \sigma_z\right) = -200 \cdot 10^3 \cdot 1.25 \cdot 10^{-5} \cdot 50 + \frac{1}{3}\left((-50) + (-215.5)\right) = -212.5\ MPa$$

$$[S] = \begin{bmatrix} -212.5 & 0 & 0 \\ 0 & -212.5 & 0 \\ 0 & 0 & -50 \end{bmatrix} (MPa)$$ - tenzor napona

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \upsilon\left(\sigma_x + \sigma_y\right)\right] + \alpha_t \Delta t = \frac{1}{200 \cdot 10^3}\left[-50 - \frac{1}{3}(-212.5 - 212.5)\right] + 1.25 \cdot 10^{-5} \cdot 50 = 1.083 \cdot 10^{-3}$$ - **<u>izduženje</u>**

$$[D]=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1.083 \end{bmatrix}\cdot 10^{-3}$$ - tenzor deformacije

b. komponentalna pomjeranja

$$\varepsilon_z=\frac{\partial w}{\partial z}=1.083\cdot 10^{-3} \quad\rightarrow\quad \partial w=1.083\cdot 10^{-3}\partial z \;\Big|\int \quad\rightarrow\quad w=1.083\cdot 10^{-3}\cdot z+C$$

Granični uslov: za z=0 w=0 $\rightarrow$ $0=1.083\cdot 10^{-3}\cdot 0+C$ $\rightarrow$ $C=0$

Konačno: $u=0,\;\; v=0,\;\; w=1.083\cdot 10^{-3}\cdot z$

c. promjena visine bloka (Δh)

$$\varepsilon_z=\frac{\Delta h}{h} \quad\rightarrow\quad \Delta h=\varepsilon_z\cdot h=1.083\cdot 10^{-3}\cdot 100\approx 0.11\; cm$$

d. vrijednost kubne dilatacije (specifična promjena zapremina)

$$e=\varepsilon_x+\varepsilon_y+\varepsilon_z=\left(0+0+1.083\right)\cdot 10^{-3}=1.083\cdot 10^{-3}\approx 11\cdot 10^{-4}$$

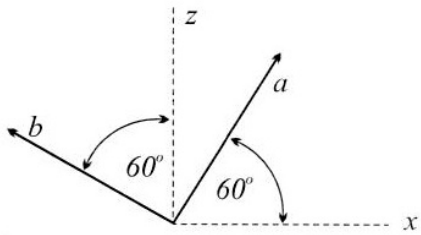
e. apsolutna promjena zapremine bloka

$$\Delta V=e\cdot V=11\cdot 10^{-4}\cdot 1\cdot 0.1\cdot 0.1=11\cdot 10^{-6}\; m^3=11\; cm^3$$

3.

ZADATAK:

U tački A napregnutog tela u kojoj vlada ravno stanje napona, merenjem dobijeni su sledeći podaci:
σ_z = 30 MPa,
σ_a=-0.98 MPa i τ_{ab} = 23.66 MPa.
Odrediti:
a/ tenzor napona, glavne napone, glavne pravce i izdvojiti element na koji deluju,
b/ pravce za koje je normalni napon jednak nuli,
c/Mohr-ov krug napona (proveriti rezultate pod a/ i b/),
d/ vrednost Poisson-ovog koeficijenta ν (ε_y = -5·10^{-5}, E=200 GPa),
e/ tenzor deformacije, glavne dilatacije i glavne pravce.



R J E Š E NJ E

a. tenzor napona i glavni naponi

$\varphi_a=60^0 \quad\rightarrow$

$$\sigma_a=\frac{1}{2}\left(\sigma_x+\sigma_z\right)+\frac{1}{2}\left(\sigma_x-\sigma_z\right)\cos 2\varphi_a+\tau_{xz}\sin 2\varphi_a=$$
$$=\frac{1}{2}\left(\sigma_x+30\right)+\frac{1}{2}\left(\sigma_x-30\right)\cos\left(2\cdot 60^0\right)+\tau_{xz}\sin\left(2\cdot 60^0\right)=-0.98\; MPa$$

$\varphi_b=150^0 \quad\rightarrow$

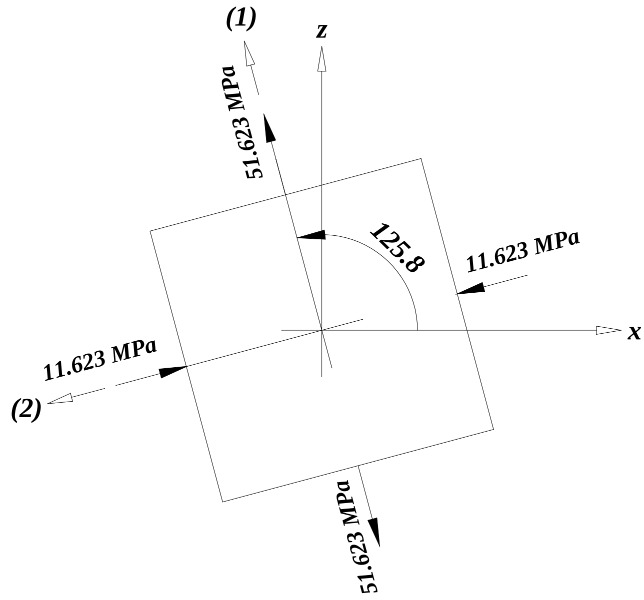
$$\tau_{ba}=\frac{1}{2}\left(\sigma_x-\sigma_z\right)\sin 2\varphi_b-\tau_{xz}\cos 2\varphi_b=$$
$$=\frac{1}{2}\left(\sigma_x-30\right)\sin\left(2\cdot 150^0\right)-\tau_{xz}\cos\left(2\cdot 150^0\right)=23.66\; MPa$$

Iz sistema jednačina se dobija: $\sigma_x=10\; MPa$, $\tau_{xz}=-30\; MPa$ $\rightarrow$ $\boxed{[S]=\begin{bmatrix} 10 & -30 \\ -30 & 30 \end{bmatrix} MPa}$ - **<u>ravno</u> stanje napona u tački**

$$\sigma_{1/2} = \frac{1}{2}(10+30) \pm \frac{1}{2}\sqrt{(10-30)^2 + 4(-30)^2} = 20 \pm 31.623$$

$$\left.\begin{array}{l}\sigma_1 = 51.623\ MPa \\ \sigma_2 = -11.623\ MPa\end{array}\right\}\ \text{- Glavni normalni naponi}$$

Kontrola $\sigma_1 + \sigma_2 = \sigma_y + \sigma_z$



$$tg2\alpha = \frac{2\cdot(-30)}{10-30} = 3$$

$$(\sigma_x - \sigma_z) < 0 \quad \rightarrow \quad \alpha = \frac{1}{2}arctg(3) + 90° = 125.8^0$$

b. pravci za koje je normalni napon jednak nula (ako postoje)

$$\sigma_n = \frac{1}{2}(\sigma_x + \sigma_z) + \frac{1}{2}(\sigma_x - \sigma_z)\cos 2\varphi + \tau_{xz}\sin 2\varphi = 0$$

$$\sigma_n = \frac{1}{2}(\sigma_1 + \sigma_2) + \frac{1}{2}(\sigma_1 - \sigma_2)\cos 2\varphi + \tau_{12}\sin 2\varphi = 0$$ (u sistemu glavnih osa $\tau_{12} = 0$)

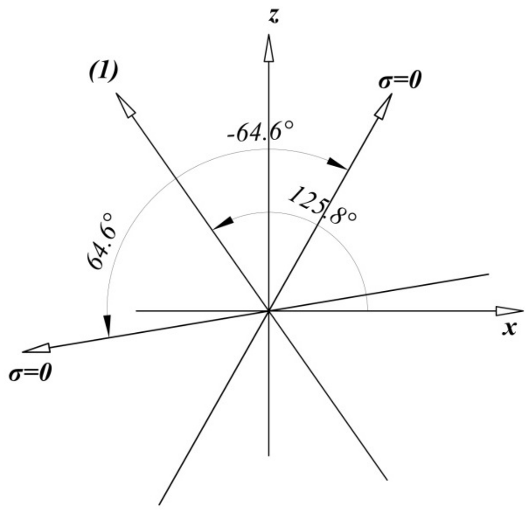
$$\sigma_n = \frac{1}{2}(51.623 - 11.623) + \frac{1}{2}(51.623 + 11.623)\cos 2\varphi = 0$$

$$20 + 31.623\cos 2\varphi = 0$$

$$\cos 2\varphi = -0.632 \quad \rightarrow \quad \textit{postoji rješenje}$$

$$2\varphi = arccoc(-0.632) = \pm 129.2^0$$

$$\varphi = \pm 64.6^0 \ \ - \textit{u odnosu na osu } (1)$$

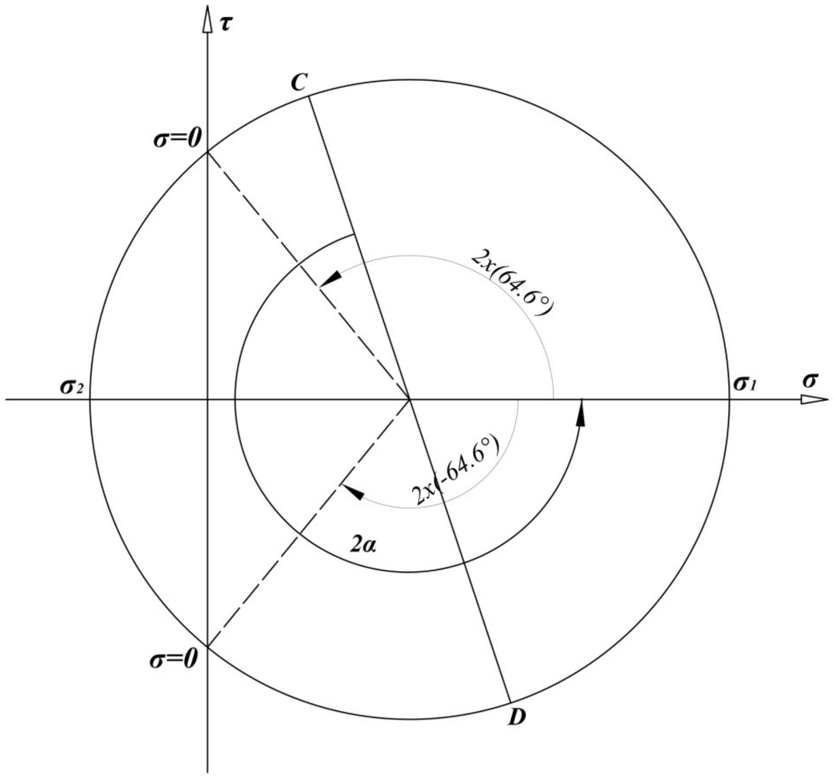


c. *Mohr*-ov krug napona

$C[\sigma_x; -\tau_{xz}] = [10;30]\ MPa$

$D[\sigma_z; \tau_{xz}] = [30;-30]\ MPa$

Usvaja se razmjer 10 *MPa* = 1 *cm*



d. *Poisson*-ov koeficijent

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \upsilon\left(\sigma_z + \sigma_x\right)\right] = -5\cdot 10^{-5}$$

$$-\upsilon\left(10+30\right) = -5\cdot 10^{-5}\cdot 200\cdot 10^{3}$$

$$\upsilon = \frac{1}{4} = 0.25$$

e. Tenzor deformacije i glavne dilatacije

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \upsilon\left(\sigma_y + \sigma_z\right)\right] = \frac{1}{200\cdot 10^{3}}\left[10 - 0.25\left(0+30\right)\right] = 1.25\cdot 10^{-5}$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \upsilon\left(\sigma_x + \sigma_y\right)\right] = \frac{1}{200\cdot 10^{3}}\left[30 - 0.25\left(10+0\right)\right] = 13.75\cdot 10^{-5}$$

$$G = \frac{E}{2\left(1+\upsilon\right)} = \frac{200}{2\left(1+0.25\right)} = 80\ GPa$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = 0$$

$$\gamma_{xz} = \frac{\tau_{xz}}{G} = \frac{-30}{80\cdot 10^{3}} = -37.5\cdot 10^{-5}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G} = 0$$

$$\left[D\right] = \begin{bmatrix} 1.25 & 0 & -18.75 \\ 0 & -5 & 0 \\ -18.75 & 0 & 13.75 \end{bmatrix}\cdot 10^{-5}$$ - **<u>prostorno</u> stanje deformacije u tački**

$$\varepsilon_{a/b} = \frac{1}{2}\left(\varepsilon_x + \varepsilon_z\right) \pm \frac{1}{2}\sqrt{\left(\varepsilon_x - \varepsilon_z\right)^2 + \gamma_{xz}^2} =$$

$$= \frac{1}{2}\left(1.25+13.75\right)\cdot 10^{-5} \pm \frac{1}{2}\cdot 10^{-4}\sqrt{\left(1.25-13.75\right)^2 + 37.5^2} = \left(7.5 \pm 19.75\right)\cdot 10^{-4}$$

$$\varepsilon_a = 27.25\cdot 10^{-5}$$

$$\varepsilon_b = -12.25\cdot 10^{-5}$$

$$\varepsilon_1 = \varepsilon_a = 27.25\cdot 10^{-5}$$

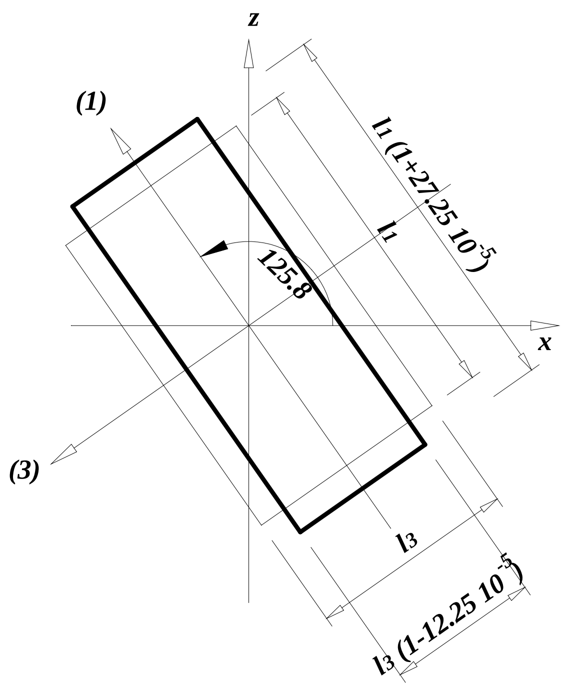
$$\varepsilon_2 = -5\cdot 10^{-5}$$

$$\varepsilon_3 = \varepsilon_b = -12.25\cdot 10^{-5}$$

$$tg2\alpha = \frac{\gamma_{xz}}{\varepsilon_x - \varepsilon_z} = \frac{-37.5\cdot 10^{-5}}{\left(1.25-13.75\right)\cdot 10^{-5}} = 3$$

$$(\varepsilon_x - \varepsilon_z) < 0 \ \rightarrow \ \alpha = \frac{1}{2}arctg\left(3\right) + 90° = 125.8^0$$

$$\boxed{\boldsymbol{\alpha_\varepsilon = \alpha_\sigma = 125.8^0}}$$



4.

ZADATAK:

Pravougaona ploča a x b = 100 x 80 cm debljine t = 3 cm leži između dve paralelne nepoktetne ravni na međusobnom rastojanju 3.001 cm. Ako se ploča zagreje za Δt = 30°C i optereti silama zatezanja P_1 = 30 kN i silama pritiska P_2 = 90 kN kao na slici, odrediti:

a/ pritisak koji ploča vrši na nepokretne ravni,

b/ tenzor napona u karakterističnoj tački,

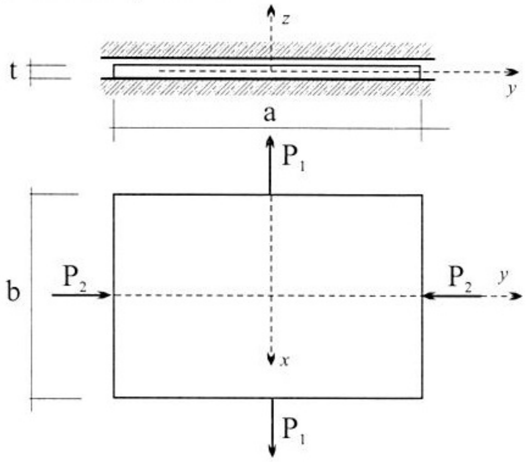
c/ tenzor deformacija u sistemu glavnih osa,

d/ specifičnu i apsolutnu promenu zapremine.

E = 2 10⁵ MPa,

ν = 0.3

α = 1.25 10⁻⁵



R J E Š E NJ E

a. pritisak koji ploča vrši na nepokretne ravni

$\boldsymbol{\sigma_x = \frac{P_1}{a\cdot t} = \frac{30}{100\cdot 3} = 0.1\frac{kN}{cm^2}}$	$\boldsymbol{\varepsilon_x \neq 0}$
$\boldsymbol{\sigma_y = -\frac{P_2}{b\cdot t} = -\frac{90}{80\cdot 3} = -0.375\frac{kN}{cm^2}}$	$\boldsymbol{\varepsilon_y \neq 0}$
$\boldsymbol{\sigma_z \neq 0}$	$\boldsymbol{\varepsilon_z = \frac{\Delta l_z}{l_z}}$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \upsilon\left(\sigma_x + \sigma_y\right)\right] + \alpha_t \Delta t = \frac{\Delta l_z}{l_z}$$

$$\frac{1}{2\cdot 10^4}\left[\sigma_z - 0.3\left(0.1 - 0.375\right)\right] + 1.25\cdot 10^{-5}\cdot 30 = \frac{0.001}{3} \quad \rightarrow \quad \sigma_z = -0.916\,\frac{kN}{cm^2}$$

$$N = \sigma_z \cdot a \cdot b = -0.916\cdot 100\cdot 80 = 7328\ kN$$

b. tenzor napona

$$\left[S\right] = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & -0.375 & 0 \\ 0 & 0 & -0.916 \end{bmatrix} \left(\frac{kN}{cm^2}\right)$$

c. tenzor deformacije

$$\varepsilon_x = \frac{1}{2\cdot 10^4}\left[0.1 - 0.3\left(-0.375 - 0.916\right)\right] + 1.25\cdot 10^{-5}\cdot 30 = 3.994\cdot 10^{-4}$$

$$\varepsilon_y = \frac{1}{2\cdot 10^4}\left[-0.375 - 0.3\left(0.1 - 0.916\right)\right] + 1.25\cdot 10^{-5}\cdot 30 = 3.685\cdot 10^{-4}$$

$$\left[D\right] = \begin{bmatrix} 3.994 & 0 & 0 \\ 0 & 3.685 & 0 \\ 0 & 0 & 3.333 \end{bmatrix} \cdot 10^{-4}$$

d. specifična i apsolutna promjena zapremine

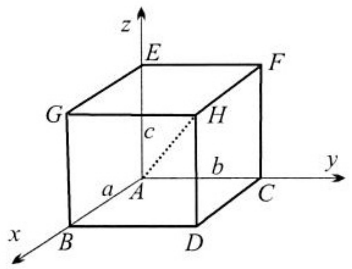
$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = (3.994 + 3.685 + 3.333) \cdot 10^{-4} = 11.012 \cdot 10^{-4}$$

$$\Delta V = e \cdot V = 11.012 \cdot 10^{-4} \cdot 100 \cdot 80 \cdot 3 = 26.429\ cm^3$$

5.

ZADATAK

Dat je kvadar čije su ivice a=4 mm, b=3 mm i c=5 mm. Posle deformacije dužine ivica su a_1=3.997 mm, b_1=3.002 mm i c_1=4.998 mm, dok ugao <CAE prelazi u $90^o 00' 35''$.



Odrediti:

a/ tenzor deformacije, glavne dilatacije, glavne pravce i prikazati deformaciju elementa u ravni yz,

b/ vrednost dilatacije u pravcu dijagonale AH, kao i promenu njene dužine,

c/ tenzor napona (E = 200 Pa, ν = 1/3), glavne napone i glavne pravce i izdvojiti element na koji deluju,

d/ komponentalne napone za ravan koja leži u yz ravni i čija normala sa x osom gradi ugao φ = -30°,

e/ Mohr-ov krug napona.

R J E Š E NJ E

a. tenzor deformacije

$$\varepsilon_x = \frac{3.997 - 4}{4} = -7.5 \cdot 10^{-4}$$

$$\varepsilon_y = \frac{3.002 - 3}{3} = 6.667 \cdot 10^{-4}$$

$$\varepsilon_z = \frac{4.998 - 5}{5} = -4 \cdot 10^{-4}$$

$$\gamma_{yz} = -35 \frac{1}{3600} \frac{\pi}{180} = -1.697 \cdot 10^{-4}$$

$$[D] = \begin{bmatrix} -7.5 & 0 & 0 \\ 0 & 6.667 & -\frac{1}{2}1.697 \\ 0 & -\frac{1}{2}1.697 & -4 \end{bmatrix} \cdot 10^{-4}$$

$$\varepsilon_1 = 6.7341 \cdot 10^{-4}$$

$$\varepsilon_2 = -4.0671 \cdot 10^{-5}$$

$$\varepsilon_3 = -7.5 \cdot 10^{-5}$$

$$\alpha = -4.52^0$$

b. dilatacija u pravcu *AH*

$$\varepsilon_{AH} = \frac{\Delta AH}{AH} = \frac{AH' - AH}{AH}$$

$$AH' \neq \sqrt{a_1^2 + b_1^2 + c_1^2}\quad uglovi\ nijesu\ pravi$$

$$\cos\alpha_{AH} = \frac{4}{\sqrt{50}};\ \cos\beta_{AH} = \frac{3}{\sqrt{50}};\ \cos\gamma_{AH} = \frac{5}{\sqrt{50}}$$

$$\varepsilon_n = \varepsilon_x \cos^2\alpha + \varepsilon_y \cos^2\beta + \varepsilon_z \cos^2\gamma + \gamma_{xy}\cos\alpha\cos\beta + \gamma_{yz}\cos\beta\cos\gamma + \gamma_{xz}\cos\alpha\cos\gamma$$

$$\varepsilon_{AH} = -7.5 \cdot 10^{-4} \cdot \left(\frac{4}{\sqrt{50}}\right)^2 + 6.667 \cdot 10^{-4} \cdot \left(\frac{3}{\sqrt{50}}\right)^2 - 4 \cdot 10^{-4} \cdot \left(\frac{5}{\sqrt{50}}\right)^2 - 1.697 \cdot 10^{-4} \cdot \frac{3}{\sqrt{50}} \cdot \frac{5}{\sqrt{50}} = -3.709 \cdot 10^{-4}$$

$$\Delta AH = -3.709 \cdot 10^{-4} \cdot AH = -3.709 \cdot 10^{-4} \cdot \sqrt{50} = -2.622 \cdot 10^{-3}\ mm$$

c. tenzor napona

$$\mu = G = \frac{E}{2(1+\upsilon)} = \frac{200 \cdot 10^3}{2\left(1 + \frac{1}{3}\right)} = 75 \cdot 10^3\ MPa$$

$$\lambda = \frac{\upsilon E}{(1+\upsilon)(1-2\upsilon)} = \frac{\frac{1}{3}200\cdot 10^3}{\left(1+\frac{1}{3}\right)\left(1-2\frac{1}{3}\right)} = 150\cdot 10^3 \ MPa$$

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = (-7.5 + 6.667 - 4)\cdot 10^{-4} = -4.833\cdot 10^{-4}$$

$$\sigma_x = 2\mu\varepsilon_x + \lambda e = 2\cdot 75\cdot 10^3\cdot\left(-7.5\cdot 10^{-4}\right) + 150\cdot 10^3\left(-4.833\cdot 10^{-4}\right) = -185 \ MPa$$

$$\sigma_y = 2\mu\varepsilon_y + \lambda e = 2\cdot 75\cdot 10^3\cdot 6.667\cdot 10^{-4} + 150\cdot 10^3\left(-4.833\cdot 10^{-4}\right) = 27.51 \ MPa$$

$$\sigma_z = 2\mu\varepsilon_z + \lambda e = 2\cdot 75\cdot 10^3\cdot\left(-4\cdot 10^{-4}\right) + 150\cdot 10^3\left(-4.833\cdot 10^{-4}\right) = -132.5 \ MPa$$

$$\tau_{xy} = \mu\gamma_{xy} = 0$$

$$\tau_{yz} = \mu\gamma_{yz} = 75\cdot 10^3\cdot\left(-1.697\cdot 10^{-4}\right) = -12.73 \ MPa$$

$$\tau_{zx} = \mu\gamma_{zx} = 0$$

$$[S] = \begin{bmatrix} -185 & 0 & 0 \\ 0 & 27.5 & -12.73 \\ 0 & -12.73 & -132.5 \end{bmatrix} (MPa)$$

d. naponi u ravni (φ=-30^0)

$$\sigma_n = -1.469 \ MPa$$

$$\tau_{nl} = -75.65 \ MPa$$

6.

ZADATAK

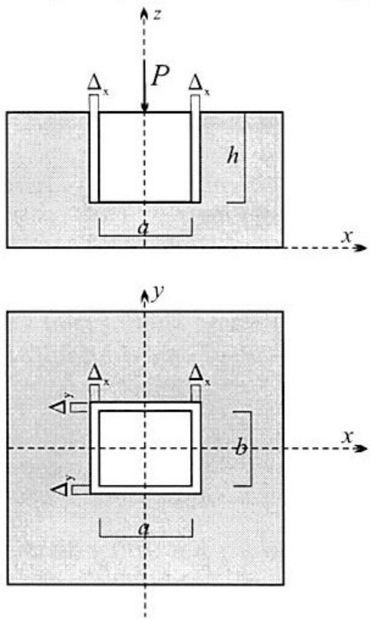
Kvadar a x b x h = 30 x 40 x 20 cm stavljen je u otvor u masivnoj ploči koju možemo smatrati nedeformabilnom. Na gornju površinu kvadra deluje sila pritiska P i kvadar se zagreva za $\Delta t = 30^o C$. Pre nanošenja opterećenja, između stranice kvadra i ivica otvora postojali su zazori u x i y pravcu $\Delta_x = 0.021$ mm i $\Delta_y = 0.054$ mm .

a/ Odrediti vrednost sile P iz uslova da se u x pravcu pojavi napon pritiska od 5 MPa.

b/ Napisati tenzor napona i deformacije u sistemu glavnih osa.

c/ Izračunati promenu dužina ivica kvadra kao i specifičnu i apsolutnu promenu zapremine.

E = 20 GPa,
ν = 0.3
α = 1.1· 10^{-5}



_R J E Š E NJ E_

a. sila P

$\boldsymbol{\sigma_x = -0.5\frac{kN}{cm^2}}$	$\boldsymbol{\varepsilon_x = \frac{2\Delta x}{a} = \frac{2\cdot 0.00021}{30} = 1.4\cdot 10^{-4}}$
$\boldsymbol{\sigma_y \neq 0}$	$\boldsymbol{\varepsilon_y = \frac{2\Delta y}{b} = \frac{2\cdot 0.00054}{40} = 2.7\cdot 10^{-4}}$
$\boldsymbol{\sigma_z = -\frac{P}{a\cdot b}}$	$\boldsymbol{\varepsilon_z \neq 0}$

$$\left.\begin{aligned}\varepsilon_x &= \frac{1}{20\cdot 10^2}\left[-0.5-0.3\left(\sigma_y+\sigma_z\right)\right]+1.1\cdot 10^{-5}\cdot 30 = 1.4\cdot 10^{-4}\\ \varepsilon_y &= \frac{1}{20\cdot 10^2}\left[\sigma_y-0.3\left(-0.5+\sigma_z\right)\right]+1.1\cdot 10^{-5}\cdot 30 = 2.7\cdot 10^{-4}\end{aligned}\right\} \rightarrow \quad \begin{aligned}\sigma_y &= -0.3\,\frac{kN}{cm^2}\\ \sigma_z &= -0.1\,\frac{kN}{cm^2}\end{aligned}$$

$$\sigma_z = -\frac{P}{a\cdot b} = -\frac{P}{40\cdot 30} = -0.1 \quad \rightarrow \quad P = 120\ kN$$

b. naponi i deformacije u sistemu glavnih osa

$$\left[S\right] = \begin{bmatrix} -0.5 & 0 & 0\\ 0 & -0.3 & 0\\ 0 & 0 & -0.1\end{bmatrix}\left(\frac{kN}{cm^2}\right)$$

$$\varepsilon_z = \frac{1}{20\cdot 10^2}\left[-0.1-0.3\left(-0.5-0.3\right)\right]+1.1\cdot 10^{-5}\cdot 30 = 4\cdot 10^{-4}$$

$$\left[D\right] = \begin{bmatrix} 1.4 & 0 & 0\\ 0 & 2.7 & 0\\ 0 & 0 & 4\end{bmatrix}\cdot 10^{-4}$$

c. promjena dužina stranica kvadra, specifična i apsolutna promjena zapremine

$$\Delta a = a\cdot\varepsilon_x = 30\cdot 1.4\cdot 10^{-4} = 4.2\cdot 10^{-3}\ cm$$

$$\Delta b = b\cdot\varepsilon_y = 40\cdot 2.7\cdot 10^{-4} = 10.8\cdot 10^{-3}\ cm$$

$$\Delta h = h\cdot\varepsilon_z = 20\cdot 4\cdot 10^{-4} = 8\cdot 10^{-3}\ cm$$

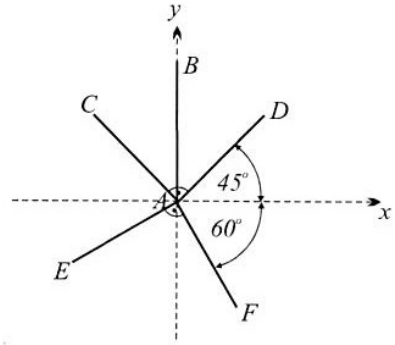
$$e = \varepsilon_x+\varepsilon_y+\varepsilon_z = \left(1.4+2.7+4\right)\cdot 10^{-4} = 8.1\cdot 10^{-4}$$

$$\Delta V = e\cdot V = 8.1\cdot 10^{-4}\cdot 20\cdot 30\cdot 40 = 19.44\ cm^3$$

7.

ZADATAK:

Na slici prikazana je duž AB od 5 cm, kao i pravi uglovi definisani oznakama <CAD i <EAF. Posle deformacije rastojanje između tačaka A i B postalo je 5.0001 cm, dok su početni pravi uglovi <CAD i <EAF prešli u $90^{\circ}00'14''$, odnosno $89^{\circ}59'52''$.



Odrediti:

a/ tenzor ravnog stanja deformacije, glavne dilatacije, glavne pravce i prikazati deformaciju elementa u ravni xy,

b/ vrednost modula elastičnosti E, ako je σx=24.667 MPa i ν=0.3,

c/ tenzor napona, glavne napone i glavne pravce i izdvojiti element na koji deluju.

R J E Š E NJ E

a. tenzor deformacije...

$$\varphi = -60^0 \quad \gamma_{nl} = \frac{8}{3600}\frac{\pi}{180} = 3.8785 \cdot 10^{-5}$$

$$\varphi = 135^0 \quad \gamma_{nl} = -\frac{14}{3600}\frac{\pi}{180} = -6.787 \cdot 10^{-5}$$

$$\varepsilon_y = \frac{5.0001-5}{5} = 2 \cdot 10^{-5}$$

$$\frac{1}{2}\gamma_{nl} = \frac{1}{2}\left(\varepsilon_x - \varepsilon_y\right)\sin 2\varphi - \frac{1}{2}\gamma_{xy}\cos 2\varphi$$

$$\left.\begin{aligned}\frac{1}{2}3.8785 \cdot 10^{-5} &= \frac{1}{2}(\varepsilon_x - 2)\cdot 10^{-5}\sin\left(2\cdot\left(-60^0\right)\right) - \frac{1}{2}\gamma_{xy}\cdot 10^{-5}\cos\left(2\cdot\left(-60^0\right)\right)\\ \frac{1}{2}\left(-6.787\cdot 10^{-5}\right) &= \frac{1}{2}(\varepsilon_x - 2)\cdot 10^{-5}\sin\left(2\cdot 135^0\right) - \frac{1}{2}\gamma_{xy}\cdot 10^{-5}\cos\left(2\cdot 135^0\right)\end{aligned}\right\} \rightarrow \begin{aligned}\varepsilon_x &= 8.787\cdot 10^{-5}\\ \frac{1}{2}\gamma_{xy} &= 9.7562\cdot 10^{-5}\end{aligned}$$

$$[D] = \begin{bmatrix}\varepsilon_x & \frac{1}{2}\gamma_{xy}\\ \frac{1}{2}\gamma_{yx} & \varepsilon_y\end{bmatrix} = \begin{bmatrix}8.787 & 9.7562\\ 9.7562 & 2\end{bmatrix}\cdot 10^{-5}$$ tenzor **ravnog stanja deformacije** u tački

$$\varepsilon_1 = 15.723\cdot 10^{-5}$$

$$\varepsilon_2 = -4.936\cdot 10^{-5}$$

$$\alpha = 35.41^0$$

b. modul elastičnosti *E*

$$\sigma_x = 2\mu\varepsilon_x + \lambda e$$

$$\mu = G = \frac{E}{2(1+\upsilon)} = \frac{E}{2(1+0.3)}$$

$$\lambda = \frac{\upsilon E}{(1+\upsilon)(1-2\upsilon)} = \frac{0.3 \cdot E}{(1+0.3)(1-2\cdot 0.3)}$$

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = (8.787 + 2)\cdot 10^{-5} = 10.787 \cdot 10^{-5}$$

$$\sigma_x = 24.667 = 2 \cdot \frac{E \cdot 10^3}{2(1+0.3)} \cdot 8.787 \cdot 10^{-5} + \frac{0.3 \cdot E \cdot 10^3}{(1+0.3)(1-2\cdot 0.3)} \cdot 10.787 \cdot 10^{-5} \ \rightarrow \ E = 190 \ GPa$$

c. tenzor napona...

$$\mu = \frac{190}{2(1+0.3)} = 73.077 \ GPa$$

$$\lambda = \frac{0.3 \cdot 190}{(1+0.3)(1-2\cdot 0.3)} = 109.615 \ GPa$$

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = (8.787 + 2)\cdot 10^{-5} = 10.787 \cdot 10^{-5}$$

$$\sigma_x = 24.667 \ MPa$$

$$\sigma_y = 2\mu\varepsilon_y + \lambda e = 2 \cdot 73.077 \cdot 10^3 \cdot 2 \cdot 10^{-5} + 109.615 \cdot 10^3 \cdot 10.787 \cdot 10^{-5} = 14.747 \ MPa$$

$$\sigma_z = 2\mu\varepsilon_z + \lambda e = 2 \cdot 73.077 \cdot 10^3 \cdot 0 + 109.615 \cdot 10^3 \cdot 10.787 \cdot 10^{-5} = 11.824 \ MPa$$

$$\tau_{xy} = \mu\gamma_{xy} = 73.077 \cdot 10^3 \cdot \left(2 \cdot 9.7562 \cdot 10^{-5}\right) = 14.259 \ MPa$$

$$\tau_{yz} = \mu\gamma_{yz} = 0$$

$$\tau_{zx} = \mu\gamma_{zx} = 0$$

$$[S] = \begin{bmatrix} 24.667 & 14.259 & 0 \\ 14.259 & 14.747 & 0 \\ 0 & 0 & 11.824 \end{bmatrix} (MPa)$$ tenzor **<u>prostornog stanja napona</u>** u tački

$$\sigma_1 = 34.804 \ MPa$$

$$\sigma_2 = 11.824 \ MPa$$

$$\sigma_3 = 4.610 \ MPa$$

$$\alpha = 35.41^0$$

$$\boxed{\boldsymbol{\alpha_\varepsilon = \alpha_\sigma = 35.41^0}}$$

8.

ZADATAK:

Telo na slici se zagreva za Δt i opterećeno je koncentrisanom silom P=350 kN..

a/ Odrediti vrednost temperature Δt ako je najveća dozvoljena pozitivna dilatacija $\varepsilon = +2.5\cdot10^{-4}$, *pod uslovom:*

1/ da se telo može slobodno deformisati,

2/ da je telo upasovano u otvor sa krutim glatkim ivicama na rastojanju $\Delta x = 2\cdot10^{-4}$ *cm.*

Zatim za oba slučaja:

b/ napisati tenzore napona i deformacija u sistemu glavnih osa,

c/.odrediti promene dimenzija bloka lx, ly i lz, kao i promenu površine porečnog preseka A,

c/ izračunati specifične i apsolutne promene zapremine.

$E = 2\cdot10^4$ kN/cm²,

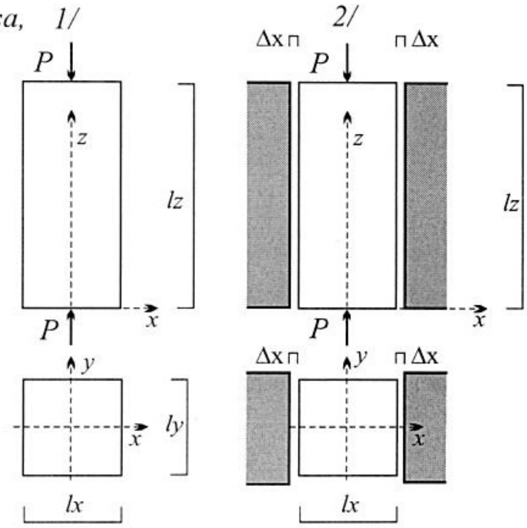
$\nu = 0.2$

$\alpha = 1.2\cdot10^{-5}$

lz = 100 cm

lx = 5 cm

ly = 7 cm



R J E Š E NJ E

<u>Slučaj 1/</u>

a. temperaturna promjena Δt

$$\boldsymbol{\sigma_x = 0} \qquad \boldsymbol{\varepsilon_x = 2.5\cdot10^{-4}}$$
$$\boldsymbol{\sigma_y = 0} \qquad \boldsymbol{\varepsilon_y = 2.5\cdot10^{-4}}$$
$$\boldsymbol{\sigma_z = -\frac{P}{l_x\cdot l_y} = -\frac{350}{5\cdot7} = -10\ \frac{kN}{cm^2}} \qquad \boldsymbol{\varepsilon_z \neq 0}$$

$$\varepsilon_y = \varepsilon_y = -\upsilon\frac{\sigma_z}{E} + \alpha\cdot\Delta t = -0.2\frac{-10}{2\cdot10^4} + 1.2\cdot10^{-5}\cdot\Delta t = 2.5\cdot10^{-4} \rightarrow \Delta t = 12.5\ ^0C$$

b. tenzori napona i deformacija

$$[S] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -10 \end{bmatrix}\left(\frac{kN}{cm^2}\right)$$

$$\varepsilon_z = \frac{\sigma_z}{E} + \alpha\cdot\Delta t = \frac{-10}{2\cdot10^4} + 1.2\cdot10^{-5}\cdot12.5 = -3.5\cdot10^{-4}$$

$$[D] = \begin{bmatrix} 2.5 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0 & 0 & -3.5 \end{bmatrix}\cdot10^{-4}$$

c. promjena dimenzija bloka i promjena površine poprečnog presjeka bloka

$$\Delta l_x = l_x\cdot\varepsilon_x = 5\cdot2.5\cdot10^{-4} = 12.5\cdot10^{-4}\ cm$$

$$\Delta l_y = l_y\cdot\varepsilon_y = 7\cdot2.5\cdot10^{-4} = 17.5\cdot10^{-4}\ cm$$

$$\Delta l_z = l_z \cdot \varepsilon_z = 100 \cdot (-3.5) \cdot 10^{-4} = -0.035\ cm$$

$$\Delta A = (\varepsilon_x + \varepsilon_y) \cdot A = (2.5 + 2.5) \cdot 10^{-4} \cdot 5 \cdot 7 = 0.0175\ cm^2$$

d. specifična i apsolutna promjena zapremine bloka

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = (2.5 + 2.5 - 3.5) \cdot 10^{-4} = 1.5 \cdot 10^{-4}$$

$$\Delta V = e \cdot V = 1.5 \cdot 10^{-4} \cdot 5 \cdot 7 \cdot 100 = 0.525\ cm^3$$

Slučaj 2/

a. temperaturna promjena Δt

$$\boldsymbol{\varepsilon_x = \frac{2\Delta x}{l_x} = \frac{2 \cdot 2 \cdot 10^{-4}}{5} = 0.8 \cdot 10^{-4} < 2.5 \cdot 10^{-4} \quad \rightarrow \quad \sigma_x \neq 0}$$

$$\boldsymbol{\varepsilon_y = 2.5 \cdot 10^{-4} \qquad \sigma_y = 0}$$

$$\boldsymbol{\varepsilon_z \neq 0 \qquad \sigma_z = -\frac{P}{l_x \cdot l_y} = -\frac{350}{5 \cdot 7} = -10\ \frac{kN}{cm^2}}$$

$$\left.\begin{aligned} \varepsilon_x &= \frac{1}{E}\left[\sigma_x - \upsilon(\sigma_y + \sigma_z)\right] + \alpha_t \Delta t = \frac{1}{2 \cdot 10^4}\left[\sigma_x - 0.2(0 - 10)\right] + 1.2 \cdot 10^{-5} \cdot \Delta t = 0.8 \cdot 10^{-4} \\ \varepsilon_y &= \frac{1}{E}\left[\sigma_y - \upsilon(\sigma_z + \sigma_x)\right] + \alpha_t \Delta t = \frac{1}{2 \cdot 10^4}\left[0 - 0.2(\sigma_x - 10)\right] + 1.2 \cdot 10^{-5} \cdot \Delta t = 2.5 \cdot 10^{-4} \end{aligned}\right\} \rightarrow \begin{aligned} \sigma_x &= -2.8333\ \frac{kN}{cm^2} \\ \Delta t &= 10.138^0 \end{aligned}$$

b. tenzori napona i deformacija

$$[S] = \begin{bmatrix} -2.833 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -10 \end{bmatrix} \left(\frac{kN}{cm^2}\right)$$

$$\varepsilon_z = \frac{1}{E}(\sigma_z - \upsilon \cdot \sigma_x) + \alpha_t \Delta t = \frac{1}{2 \cdot 10^4}\left[-10 - 0.2 \cdot (-2.8333)\right] + 1.2 \cdot 10^{-5} \cdot 10.138 = -3.5 \cdot 10^{-4}$$

$$[D] = \begin{bmatrix} 0.8 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0 & 0 & -3.5 \end{bmatrix} \cdot 10^{-4}$$

c. promjena dimenzija bloka i promjena površine poprečnog presjeka bloka

$$\Delta l_x = l_x \cdot \varepsilon_x = 5 \cdot 0.8 \cdot 10^{-4} = 4 \cdot 10^{-4}\ cm$$

$$\Delta l_y = l_y \cdot \varepsilon_y = 7 \cdot 2.5 \cdot 10^{-4} = 17.5 \cdot 10^{-4}\ cm$$

$$\Delta l_z = l_z \cdot \varepsilon_z = 100 \cdot (-3.5) \cdot 10^{-4} = -0.035\ cm$$

$$\Delta A = (\varepsilon_x + \varepsilon_y) \cdot A = (0.8 + 2.5) \cdot 10^{-4} \cdot 5 \cdot 7 = 0.01155\ cm^2$$

d. specifična i apsolutna promjena zapremine bloka

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = (0.8 + 2.5 - 3.5) \cdot 10^{-4} = -0.2 \cdot 10^{-4}$$

$$\Delta V = e \cdot V = (-0.2) \cdot 10^{-4} \cdot 5 \cdot 7 \cdot 100 = -0.07\ cm^3$$

9.

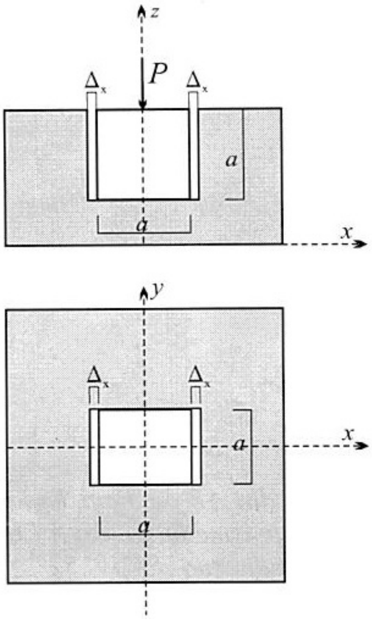
ZADATAK:

Kocka stranice a = 25 cm stavljena je u otvor u masivnoj ploči koju možemo smatrati nedeformabilnom. Na gornju površinu kocke deluje sila pritiska P i kocka se zagreva za $\Delta t = 15^o C$. Pre nanošenja opterećenja, između stranice kocke i ivica otvora postojao je u x pravcu zazor $\Delta_x = 1.5\ 10^{-3}$ cm.

a/ Odrediti vrednost sile P iz uslova da se u x pravcu pojavi napon pritiska od 1 kN/cm².

b/ Napisati tenzor napona i deformacije u sistemu glavnih osa.

c/ Izračunati specifičnu i apsolutnu promenu zapremine.



$E = 10^4\ kN/cm^2$,
$\nu = 0.2$
$\alpha = 1.1 \cdot 10^{-5}$

R J E Š E NJ E

a. sila *P*

$\boldsymbol{\sigma_x = -1\frac{kN}{cm^2}}$	$\boldsymbol{\varepsilon_x = \frac{2\Delta x}{a} = \frac{2\cdot 1.5\cdot 10^{-3}}{25} = 1.2\cdot 10^{-4}}$
$\boldsymbol{\sigma_y \neq 0}$	$\boldsymbol{\varepsilon_y = 0}$
$\boldsymbol{\sigma_z = -\frac{P}{a^2}}$	$\boldsymbol{\varepsilon_z \neq 0}$

$$\left.\begin{aligned}\varepsilon_x &= \frac{1}{10^4}\left[-1-0.2\left(\sigma_y+\sigma_z\right)\right]+1.1\cdot 10^{-5}\cdot 15 = 1.2\cdot 10^{-4}\\ \varepsilon_y &= \frac{1}{10^4}\left[\sigma_y-0.2\left(-1+\sigma_z\right)\right]+1.1\cdot 10^{-5}\cdot 15 = 0\end{aligned}\right\} \rightarrow \begin{aligned}\sigma_y &= -2\ \frac{kN}{cm^2}\\ \sigma_z &= -0.75\ \frac{kN}{cm^2}\end{aligned}$$

$$\sigma_z = -\frac{P}{a\cdot a} = -\frac{P}{25\cdot 25} = -0.75 \quad \rightarrow \quad P = 468.75\ kN$$

b. naponi i deformacije u sistemu glavnih osa

$$[S] = \begin{bmatrix} -1 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -0.75\end{bmatrix}\left(\frac{kN}{cm^2}\right)$$ - prostorno stanje napona

$$\varepsilon_z = \frac{1}{10^4}\left[-0.75-0.2\left(-1-2\right)\right]+1.1\cdot 10^{-5}\cdot 15 = 1.5\cdot 10^{-4}$$

$$[D] = \begin{bmatrix} 1.2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1.5\end{bmatrix}\cdot 10^{-4}$$ - ravno stanje deformacije

c. specifična i apsolutna promjena zapremine

$$e=\varepsilon_x+\varepsilon_y+\varepsilon_z=(1.2+1.5)\cdot 10^{-4}=2.7\cdot 10^{-4}$$

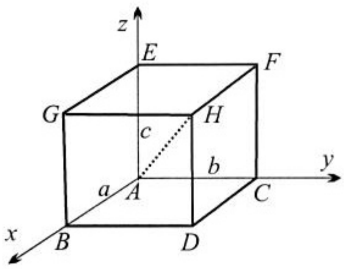
$$\Delta V=e\cdot V=2.7\cdot 10^{-4}\cdot 25^3=4.22\ cm^3$$

10.

ZADATAK

a/ Dat je kvadar čije su ivice a=4 mm, b=2 mm i c=3 mm. Posle deformacije dužine ivica su a_1=4.004 mm, b_1=2.0006 mm i c_1=2.9985 mm. Uglovi ostaju nepromenjeni.

b/ Dat je kvadar čije dužine ivica stoje u odnosu a : b : c = 2 : 3 : 5. Kvadar se tako deformiše da dužine ivica ostaju nepromenjene, a pravi uglovi prelaze u:



< BAC = 90°00'03"
< CAE = 89°59'54"
< EAB = 89°59'56"

Za dva naznačena slučaja:

1/ Napisati tenzore napona i deformacije ,

2/ Odredoto vrednost normalnog napona o dilatacije u pravcu dijagonale AH.

R J E Š E NJ E

<u>Slučaj 1/</u>

a. tenzori napona i deformacija

$$\varepsilon_x=\frac{4.004-4}{4}=1\cdot 10^{-3}$$

$$\varepsilon_y=\frac{2.0006-2}{2}=0.3\cdot 10^{-3} \qquad \rightarrow \qquad [D]=\begin{bmatrix}1 & 0 & 0\\ 0 & 0.3 & 0\\ 0 & 0 & -0.5\end{bmatrix}\cdot 10^{-3}$$

$$\varepsilon_z=\frac{2.9985-3}{3}=-0.5\cdot 10^{-3}$$

$$\mu=G=\frac{E}{2(1+\upsilon)}=\frac{200\cdot 10^3}{2\left(1+\frac{1}{3}\right)}=75\cdot 10^3\ MPa$$

$$\lambda=\frac{\upsilon E}{(1+\upsilon)(1-2\upsilon)}=\frac{\frac{1}{3}200\cdot 10^3}{\left(1+\frac{1}{3}\right)\left(1-2\frac{1}{3}\right)}=150\cdot 10^3\ MPa$$

$$e=\varepsilon_x+\varepsilon_y+\varepsilon_z=(1+0.3-0.5)\cdot 10^{-3}=0.8\cdot 10^{-3}$$

$$\sigma_x=2\mu\varepsilon_x+\lambda e=2\cdot 75\cdot 10^3\cdot 1\cdot 10^{-3}+150\cdot 10^3\cdot 0.8\cdot 10^{-3}=270\ MPa$$

$$\sigma_y=2\mu\varepsilon_y+\lambda e=2\cdot 75\cdot 10^3\cdot 0.3\cdot 10^{-3}+150\cdot 10^3\cdot 0.8\cdot 10^{-3}=165\ MPa$$

$$\sigma_z=2\mu\varepsilon_z+\lambda e=2\cdot 75\cdot 10^3\cdot\left(-0.5\cdot 10^{-3}\right)+150\cdot 10^3\cdot 0.8\cdot 10^{-3}=45\ MPa$$

$$\tau_{xy} = \mu\gamma_{xy} = 0$$
$$\tau_{yz} = \mu\gamma_{yz} = 0$$
$$\tau_{zx} = \mu\gamma_{zx} = 0$$

$$[S] = \begin{bmatrix} 270 & 0 & 0 \\ 0 & 165 & 0 \\ 0 & 0 & 45 \end{bmatrix} (MPa)$$

b. normalni napon i dilatacija u pravcu AH

$$\varepsilon_{AH} = \frac{\Delta AH}{AH} = \frac{AH' - AH}{AH}$$
$$AH = \sqrt{4^2 + 2^2 + 3^2} = \sqrt{29}$$
$$AH' = \sqrt{4.004^2 + 2.0006^2 + 2.9985^2} = \sqrt{29.0254186}$$
$$\varepsilon_{AH} = \frac{\sqrt{29.0254186} - \sqrt{29}}{\sqrt{29}} = 0.438 \cdot 10^{-3}$$

$$\cos\alpha_{AH} = \frac{4}{\sqrt{29}};\ \cos\beta_{AH} = \frac{2}{\sqrt{29}};\ \cos\gamma_{AH} = \frac{3}{\sqrt{29}}$$

$$\varepsilon_n = \varepsilon_x \cos^2\alpha + \varepsilon_y \cos^2\beta + \varepsilon_z \cos^2\gamma + \gamma_{xy}\cos\alpha\cos\beta + \gamma_{yz}\cos\beta\cos\gamma + \gamma_{xz}\cos\alpha\cos\gamma$$
$$\varepsilon_{AH} = 1 \cdot 10^{-3} \cdot \left(\frac{4}{\sqrt{29}}\right)^2 + 0.3 \cdot 10^{-3} \cdot \left(\frac{2}{\sqrt{29}}\right)^2 - 0.5 \cdot 10^{-3} \cdot \left(\frac{3}{\sqrt{29}}\right)^2 = 0.438 \cdot 10^{-3}$$

$$\sigma_n = \sigma_x \cos^2\alpha + \sigma_y \cos^2\beta + \sigma_z \cos^2\gamma + 2\tau_{xy}\cos\alpha\cos\beta + 2\tau_{yz}\cos\beta\cos\gamma + 2\tau_{xz}\cos\alpha\cos\gamma$$
$$\sigma_{AH} = 270 \cdot \left(\frac{4}{\sqrt{29}}\right)^2 + 165 \cdot \left(\frac{2}{\sqrt{29}}\right)^2 + 45 \cdot \left(\frac{3}{\sqrt{29}}\right)^2 = 214.14\ MPa$$

Slučaj 2/

a. tenzori napona i deformacija

$(\mu = 75 \cdot 10^3\ MPa)$

$$\gamma_{xy} = -3'' = -\frac{3}{3600}\frac{\pi}{180} rad = -0.145 \cdot 10^{-4}$$
$$\gamma_{yz} = 6'' = \frac{6}{3600}\frac{\pi}{180} rad = 0.294 \cdot 10^{-4}$$
$$\gamma_{xz} = 4'' = \frac{4}{3600}\frac{\pi}{180} rad = 0.194 \cdot 10^{-4}$$

$\rightarrow$

$$\tau_{xy} = \mu\gamma_{xy} = -1.0875\ MPa$$
$$\tau_{yz} = \mu\gamma_{yz} = 2.1828\ MPa$$
$$\tau_{zx} = \mu\gamma_{zx} = 1.455\ MPa$$

$$[D] = \begin{bmatrix} 0 & -0.725 & 0.97 \\ -0.725 & 0 & 1.455 \\ 0.97 & 1.455 & 0 \end{bmatrix} \cdot 10^{-5}$$

$$[S] = \begin{bmatrix} 0 & -1.0875 & 1.455 \\ -1.0875 & 0 & 2.1825 \\ 1.455 & 2.1825 & 0 \end{bmatrix} (MPa)$$

b. normalni napon i dilatacija u pravcu *AH*

$$\cos\alpha_{AH} = \frac{4}{\sqrt{29}};\ \cos\beta_{AH} = \frac{2}{\sqrt{29}};\ \cos\gamma_{AH} = \frac{3}{\sqrt{29}}$$

$$\varepsilon_n = \varepsilon_x \cos^2\alpha + \varepsilon_y \cos^2\beta + \varepsilon_z \cos^2\gamma + \gamma_{xy}\cos\alpha\cos\beta + \gamma_{yz}\cos\beta\cos\gamma + \gamma_{xz}\cos\alpha\cos\gamma$$

$$\varepsilon_{AH} = (-0.145)\cdot 10^{-4}\cdot\frac{4}{\sqrt{29}}\frac{2}{\sqrt{29}} + 0.291\cdot 10^{-4}\cdot\frac{2}{\sqrt{29}}\frac{3}{\sqrt{29}} + 0.194\cdot 10^{-4}\cdot\frac{4}{\sqrt{29}}\frac{3}{\sqrt{29}} = 0.1\cdot 10^{-3}$$

$$\sigma_n = \sigma_x \cos^2\alpha + \sigma_y \cos^2\beta + \sigma_z \cos^2\gamma + 2\tau_{xy}\cos\alpha\cos\beta + 2\tau_{yz}\cos\beta\cos\gamma + 2\tau_{xz}\cos\alpha\cos\gamma$$

$$\sigma_{AH} = 2\cdot(-1.0875)\cdot\frac{4}{\sqrt{29}}\frac{2}{\sqrt{29}} + 2\cdot 2.1825\cdot\frac{2}{\sqrt{29}}\frac{3}{\sqrt{29}} + 2\cdot 1.455\cdot\frac{4}{\sqrt{29}}\frac{3}{\sqrt{29}} = 1.5072\ MPa$$

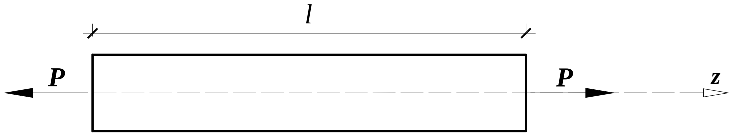
VJEŽBA BR. 9

AKSIJALNO NAPREZANJE I

1. Štep dužina l=1m, pravougaonog poprečnog presjeka dimenzija 4 x 5cm opterećen je silom zatezanja P=250 kN. (E=20 MN/cm^2; ν=0.3)

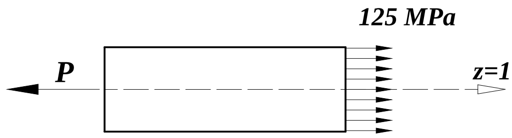
Odrediti:

a. Napone i deformaciju štapa i ukupno izduženje štapa;
b. Promjenu zapremine i površine poprečnog presjeka štapa;
c. Komponentalne napone u tačkama koje se nalaze u presječnoj ravni štapa koja zaklapa ugao 60^0 sa osom z;
d. Maksimalni smičući napon i ravan u kojoj djeluje;
e. *Mohr*-ov krug napona.



R J E Š E NJ E

a. Naponi:



$$\sigma_z = \frac{P}{A} = \frac{250}{4 \cdot 5} = 12.5 \, \frac{kN}{cm^2} \qquad [S] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 12.5 \end{bmatrix} \left[\frac{kN}{cm^2} \right] \text{ - Tenzor napona}$$

Deformacije:

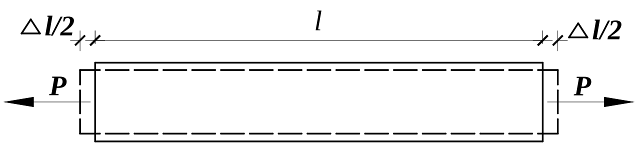
$$\varepsilon_z = \frac{\sigma_z}{E} = \frac{12.5}{20 \cdot 10^3} = 6.25 \cdot 10^{-4}$$

$$\varepsilon_x = \varepsilon_y = -\nu \cdot \varepsilon_z = -0.3 \cdot 6.25 \cdot 10^{-4} = -1.875 \cdot 10^{-4}$$

$$[D] = \begin{bmatrix} -1.875 & 0 & 0 \\ 0 & -1.875 & 0 \\ 0 & 0 & 6.25 \end{bmatrix} \cdot 10^{-4} \text{ - Tenzor deformacije}$$

Ukupno izduženje štapa:

$$\Delta l = \frac{P \cdot l}{E \cdot A} = \frac{250 \cdot 100}{20 \cdot 10^3 \cdot 20} = 0.0625 \, cm \quad \text{ili} \quad \Delta l = \varepsilon_z \cdot l = 6.25 \cdot 10^{-4} \cdot 100 = 0.0625 \, cm$$



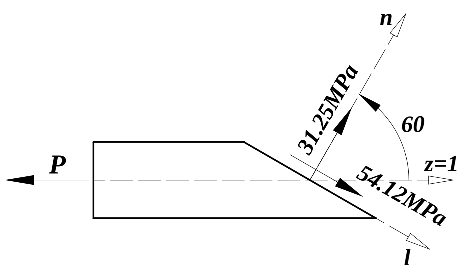
b. Promjena zapremine:

$$\Delta V = e \cdot V = \left(\varepsilon_x + \varepsilon_y + \varepsilon_z \right) \cdot V = \left(-1.875 - 1.875 + 6.25 \right) \cdot 10^{-4} \cdot 4 \cdot 5 \cdot 100 = 0.5 \, cm^3 - povećala\ se\ zapre\min$$

Promjena površine poprečnog presjeka:

$$\Delta A = \left(\varepsilon_x + \varepsilon_y \right) \cdot A = \left(-1.875 - 1.875 \right) \cdot 10^{-4} \cdot 4 \cdot 5 = -0.0075 \, cm^2 \ - smanjila\ se\ površina\ pp$$

c.



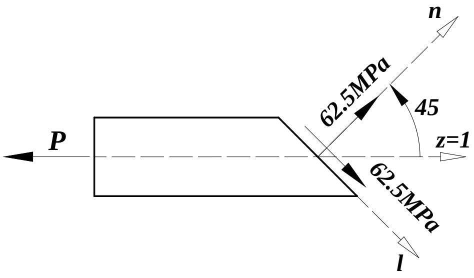
$$\sigma_n = \frac{1}{2} \sigma_z \left(1 + \cos 2\varphi \right) = \frac{1}{2} 12.5 \left(1 + \cos 2 \cdot 60^0 \right) = 3.125 \, \frac{kN}{cm^2}$$

$$\tau_{nl} = \frac{1}{2} \sigma_z \sin 2\varphi = \frac{1}{2} 12.5 \sin 2 \cdot 60^0 = 5.412 \, \frac{kN}{cm^2}$$

d.

$$\tau_{\max} = \frac{1}{2}\sigma_z = \frac{1}{2}12.5 = 6.25\ \frac{kN}{cm^2}$$

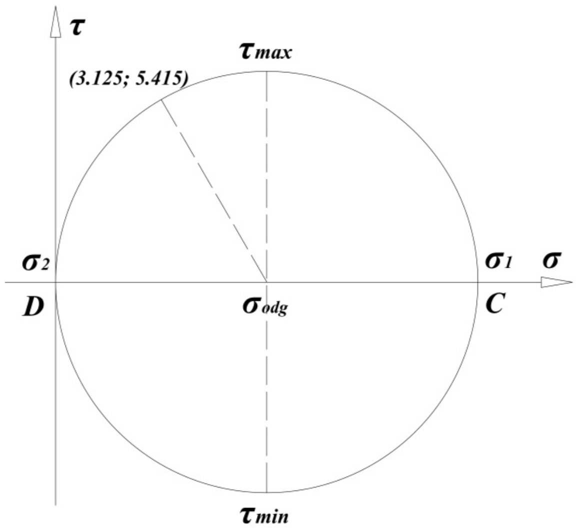
$$\sigma_{odg} = \frac{1}{2}\sigma_z = \frac{1}{2}12.5 = 6.25\ \frac{kN}{cm^2}$$



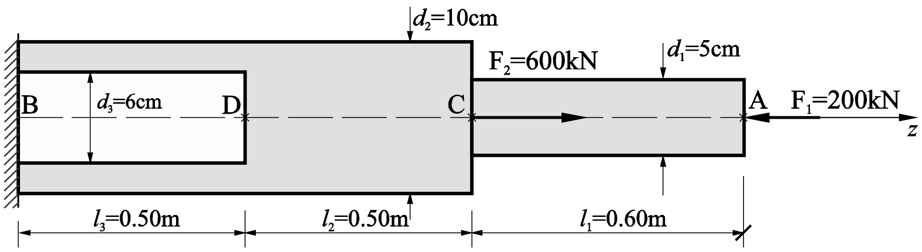
e.

$$C\left[\sigma_z; -\tau_{yz}\right] = \left[12.5;\ 0\right]\ \frac{kN}{cm^2}$$

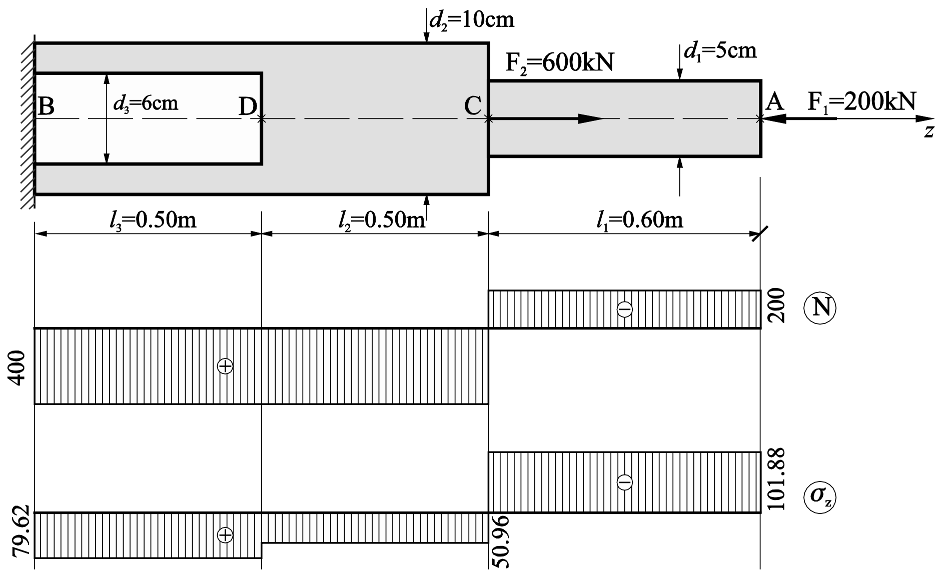
$$D\left[\sigma_y; \tau_{yz}\right] = \left[0;\ 0\right]\ \frac{kN}{cm^2}$$



2. Za nosač na slici nacrtati dijagram normalnih napona duž štapa, sračunati ukupnu promjenu dužine štapa i nacrtati dijagram normalnih napona u presjecima ***B*** i C_{desno}. Štap je izrađen od čelika sa modulom elastičnosti $E = 2.1 \cdot 10^5$ ***MPa***. Štap je skokovitog kružnog i prstenastog poprečnog presjeka sa prečnicima presjeka datim na slici.



R J E Š E NJ E



Dijagram normalnih sila - presječne normalne sile u svim karakterističnim presjecima štapa:

Dio *AC*: $N_{AC} = -F_1 = -200\ kN$

Dio *CD*: $N_{CD} = -F_1 + F_2 = 400\ kN$

Dio *DB*: $N_{DB} = N_{CD} = 400\ kN$

Površine poprečnih presjeka:

Dio *AC*: $A_{AC} = \dfrac{d_1^2 \pi}{4} = \dfrac{5^2 \pi}{4} = 19.63\ cm^2$

Dio *CD*: $A_{CD} = \dfrac{d_2^2 \pi}{4} = \dfrac{10^2 \pi}{4} = 78.50\ cm^2$

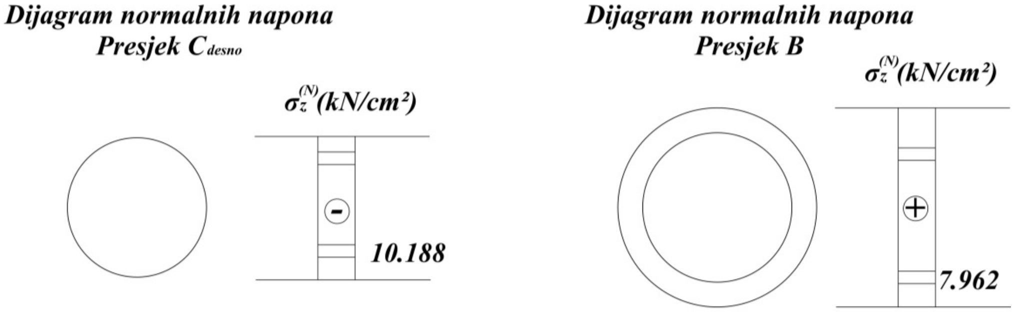
Dio *DB*: $A_{DB} = \dfrac{d_2^2 \pi}{4} - \dfrac{d_3^2 \pi}{4} = \dfrac{10^2 \pi}{4} - \dfrac{6^2 \pi}{4} = 50.24\ cm^2$

Naponi u karakterističnim poprečnim presjecima štapa:

Dio *AC* (Presjeci *A* i *C*$_{desno}$): $\sigma_{AC} = \dfrac{N_{AC}}{A_{AC}} = -\dfrac{200}{19.63} = -10.188\ \dfrac{kN}{cm^2}$

Dio *CD* (Presjeci *C*$_{lijevo}$ i *D*$_{desno}$): $\sigma_{CD} = \dfrac{N_{CD}}{A_{CD}} = \dfrac{400}{78.50} = 5.096 \dfrac{kN}{cm^2}$

Dio *DB* (Presjeci *B* i *D*$_{lijevo}$): $\sigma_{DB} = \dfrac{N_{DB}}{A_{DB}} = \dfrac{400}{50.24} = 7.962\ \dfrac{kN}{cm^2}$



Ukupnu promjena dužine štapa:

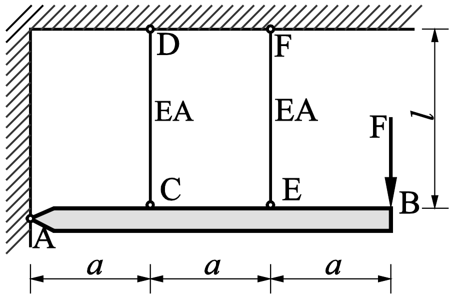
$$\Delta l = \sum_{i=1}^{n} \frac{N_i l_i}{EA_i}$$

$$\Delta l_{AB} = \Delta l_{AC} + \Delta l_{CD} + \Delta l_{DB}$$

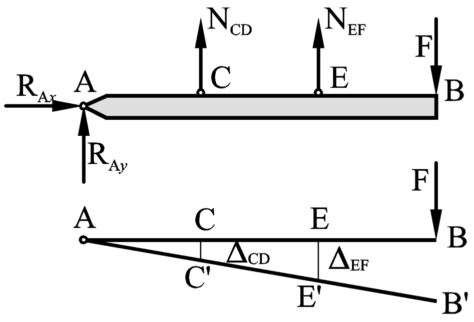
$$\Delta l_{AB} = \frac{N_{AC}\, l_{AC}}{EA_{AC}} + \frac{N_{CD}\, l_{CD}}{EA_{CD}} + \frac{N_{DB}\, l_{DB}}{EA_{DB}} =$$

$$= -\frac{200 \cdot 0.60 \cdot 10^2}{2.1 \cdot 10^4 \cdot 19.63} + \frac{400 \cdot 0.50 \cdot 10^2}{2.1 \cdot 10^4 \cdot 78.50} + \frac{400 \cdot 0.50 \cdot 10^2}{2.1 \cdot 10^4 \cdot 50.24} = -0.0292 + 0.0121 + 0.0190 = 0.0019\ cm$$

3. Apsolutno kruta greda ***AB*** u tački ***A*** je vezana zglobno za vertikalni zid i sa dva čelična štapa istog poprečnog presjeka za plafon. Ako na gredu ***AB*** u tački ***B*** djeluje vertikalna sila ***F*** odrediti reakcije veza.



R J E Š E NJ E



Uslovi ravnoteže:

$$\Sigma M^A = 0 \qquad N_{CD} \cdot a + N_{EF} \cdot 2a - F \cdot 3a = 0$$

$$\Sigma X = 0 \qquad R_{Ax} = 0$$

$$\Sigma Y = 0 \qquad R_{Ay} + N_{CD} + N_{EF} - F = 0$$

Deformacijski uslov:

$$\Delta_{CD} : a = \Delta_{EF} : 2a \quad \rightarrow \quad \Delta_{EF} = 2\Delta_{CD} \quad \rightarrow \quad \frac{N_{EF}\, l}{EA} = \frac{2N_{CD}\, l}{EA} \quad \rightarrow \quad N_{EF} = 2N_{CD}$$

Konačno, iz uslova ravnoteže dobijamo:

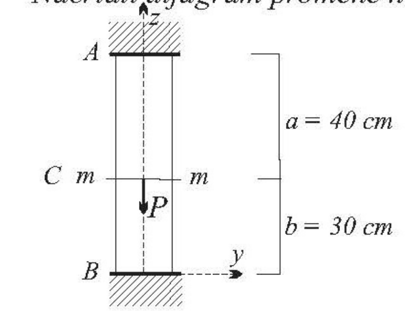
$$N_{CD} + 2N_{CD} \cdot 2 - F \cdot 3 = 0 \quad \rightarrow \quad N_{CD} = \frac{3}{5}F$$

$$N_{EF} = \frac{6}{5}F$$

$$R_{Ay} = -N_{CD} - N_{EF} + F = -\frac{3}{5}F - \frac{6}{5}F + F = -\frac{4}{5}F$$

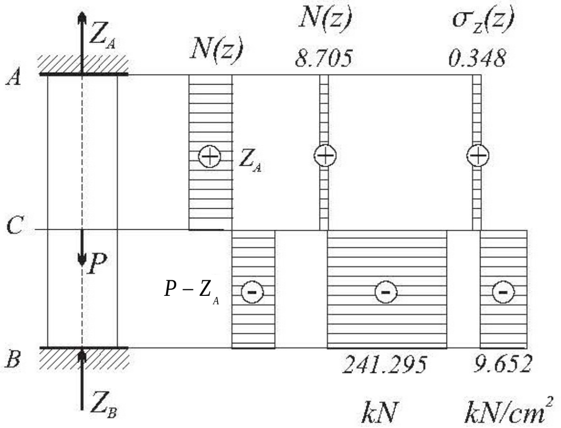
4.

Obostrano uklješten čelični štap opterećen je aksijalnom silom P u preseku m-m i zagrejan za 15° C. Nacrtati dijagram promene normalne sile i napona duž ose štapa i odrediti položaj preseka m-m.



$E = 210\ GPa$
$\alpha = 12.5 \cdot 10^{-6}\ C^{-1}$
$A = 25\ cm^2$
$P = 250\ kN$

Rešenje:



uslov ravnoteže $\quad \Sigma Z = 0 \quad Z_A + Z_B = P$

deform. uslov $\quad \Delta l = 0$

Deformacijski uslov napisan u razvijenom obliku

$$\Delta l = \Delta l_{ac} + \Delta l_{cb} =$$

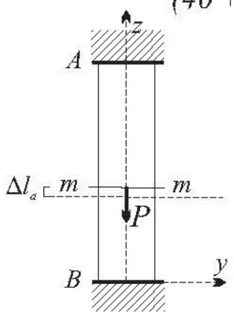
$$\underbrace{\left(\frac{Z_A a}{EA} + \alpha\,\Delta t\, a\right)}_{\Delta l_{ac}} + \underbrace{\left(\frac{(Z_A - P)b}{EA} + \alpha\,\Delta t\, b\right)}_{\Delta l_{cb}} = 0$$

$$Z_A = \frac{EA}{(a+b)}\left[\frac{P\,b}{EA} - \alpha\,\Delta t\,(a+b)\right] =$$

$$\frac{210 \cdot 10^2 \cdot 25}{(40+30)}\left[\frac{250\ 30}{210 \cdot 10^2 \cdot 25} - 12.5 \cdot 10^{-6} \cdot 15 \cdot 70\right] = 8.705\ kN$$

$Z_A = 8.705\ kN$ (zatezanje)
$Z_B = 241.295\ kN$ (pritisak)

$\sigma_A = Z_A/A = 8.705/25 = 0.348\ kN/cm^2$
$\sigma_B = Z_B/A = -241.295/25 = -9.652\ kN/cm^2$



Deo AC doživljava izduženje

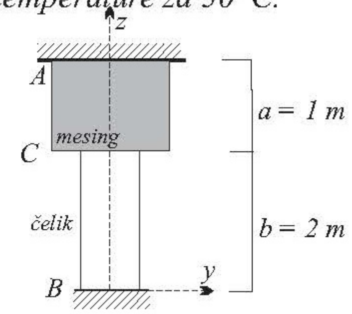
$$\Delta l_{ac} = \frac{Z_A a}{EA} + \alpha\,\Delta t\, a =$$

$$= \frac{8.705 \cdot 40}{210 \cdot 10^2 \cdot 25} + 12.5 \cdot 10^{-6} \cdot 15 \cdot 40 = 0.00816\ cm$$

za istu vrednost deo CB doživljava skraćenje

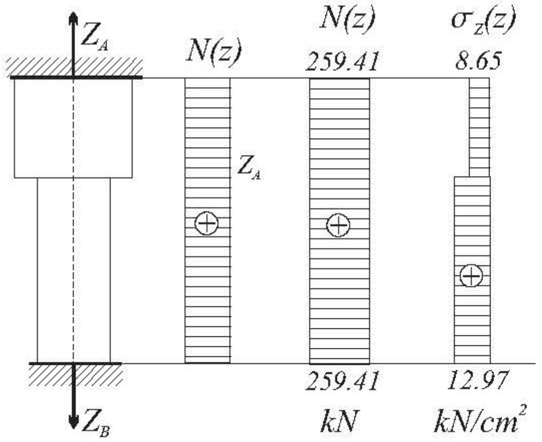
5.

Dat je štap delimično od čelika, a delimično od mesinga. Odrediti normalni napon pri smanjenju temperature za 50° C.



$E_č = 210\ GPa \qquad E_m = 100\ GPa$
$\alpha_č = 12.5 \cdot 10^{-6}\ C^{-1} \qquad \alpha_m = 17 \cdot 10^{-6}\ C^{-1}$
$A_č = 20\ cm^2 \qquad A_m = 30\ cm^2$

Rešenje:



uslov ravnoteže $\quad \Sigma Z = 0 \quad Z_A - Z_B = 0$

deform. uslov $\quad \Delta l = 0$

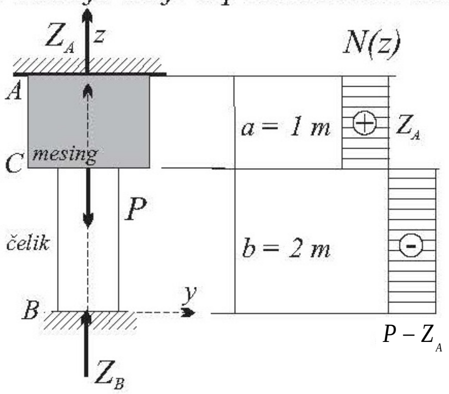
Deformacijski uslov napisan u razvijenom obliku

$$\Delta l = \Delta l_{ac} + \Delta l_{cb} =$$

$$\underbrace{\left(\frac{Z_A a}{E_m A_m} - \alpha_m\,\Delta t\, a\right)}_{\Delta l_{ac}} + \underbrace{\left(\frac{Z_A b}{E_č A_č} - \alpha_č\,\Delta t\, b\right)}_{\Delta l_{cb}} = 0$$

$Z_A = Z_B = 259.41\ kN$
$\sigma_A = Z_A/A_m = 259.41/30 = 8.65\ kN/cm^2$
$\sigma_B = Z_B/A_č = 259.41/20 = 12.97 kN/cm^2$

U slučaju da je u prethodnom zadatku bila zadata i sila:



uslov ravnoteže $\quad \Sigma Z = 0 \quad Z_A + Z_B = P$

deform. uslov $\quad \Delta l = 0$

Deformacijski uslov napisan u razvijenom obliku

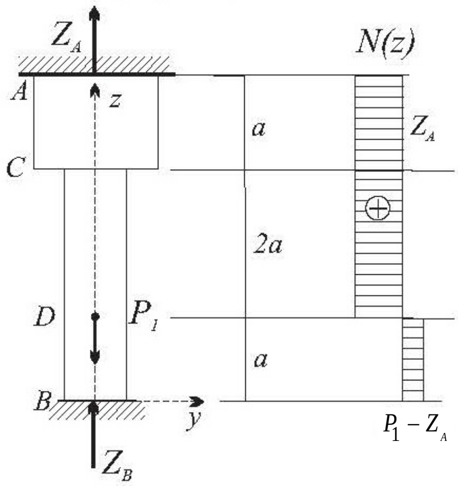
$$\Delta l = \Delta l_{ac} + \Delta l_{cb} =$$

$$\left(\frac{Z_A a}{E_m A_m} - \alpha_m \Delta t\, a\right) + \left(\frac{(Z_A - P)b}{E_č A_č} - \alpha_č \Delta t\, b\right) = 0$$

znak minus je zbog smanjenja temperature

6.

Obostrano uklješten čelični štap opterećen je aksijalnom silom P = 50 kN, kao na slici. Nacrtati dijagram promene normalne sile i napona duž ose štapa i odrediti položaj preseka C i D.



$E = 210$ GPa $\quad a = 1$ m

$A_{AC} = 15\ cm^2 \quad A_{CB} = 10\ cm^2$

uslov ravnoteže $\quad \Sigma Z = 0 \quad Z_A + Z_B - P_1 = 0$

deform. uslov $\quad \Delta l = 0$

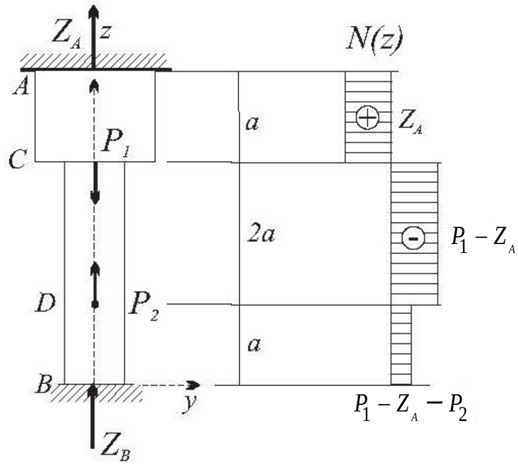
Deformacijski uslov napisan u razvijenom obliku

$$\Delta l = \Delta l_{ac} + \Delta l_{cd} + \Delta l_{db} =$$

$$\underbrace{\frac{Z_A a}{EA_{AC}}}_{\Delta l_{ac}} + \underbrace{\frac{Z_A 2a}{EA_{CB}}}_{\Delta l_{cd}} + \underbrace{\frac{(Z_A - P_1)a}{EA_{CB}}}_{\Delta l_{db}} = 0$$

7.

Obostrano uklješten čelični štap opterećen je aksijalnim silama $P_1 = 20$ kN i $P_2 = 5$ kN, kao na slici. Nacrtati dijagram promene normalne sile i napona duž ose štapa i odrediti položaj preseka C i D.



$E = 210$ GPa $\quad a = 1$ m

$A_{AC} = 15\ cm^2 \quad A_{CB} = 10\ cm^2$

uslov ravnoteže $\quad \Sigma Z = 0 \quad Z_A + Z_B - P_1 + P_2 = 0$

deform. uslov $\quad \Delta l = 0$

Deformacijski uslov napisan u razvijenom obliku

$$\Delta l = \Delta l_{ac} + \Delta l_{cd} + \Delta l_{db} =$$

$$\underbrace{\frac{Z_A a}{EA_{AC}}}_{\Delta l_{ac}} + \underbrace{\frac{(Z_A - P_1)2a}{EA_{CB}}}_{\Delta l_{cd}} + \underbrace{\frac{(Z_A - P_1 + P_2)a}{EA_{CB}}}_{\Delta l_{db}} = 0$$

U slučaju da je u prethodnom zadatku postojalo i zagrevanje dela CB deformacijski uslov bi glasio:

$$\Delta l = \Delta l_{ac} + \Delta l_{cd} + \Delta l_{db} =$$

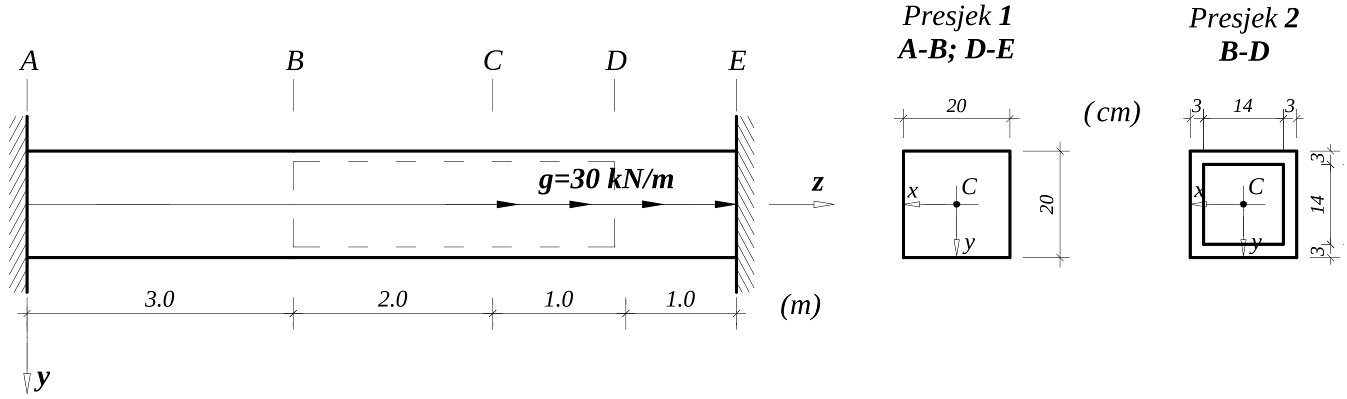
$$\underbrace{\frac{Z_A a}{EA_{AC}}}_{\Delta l_{ac}} + \underbrace{\left[\frac{(Z_A - P_1)2a}{EA_{CB}} + \alpha \Delta t\, 2a\right]}_{\Delta l_{cd}} + \underbrace{\left[\frac{(Z_A - P_1 + P_2)a}{EA_{CB}} + \alpha \Delta t\, a\right]}_{\Delta l_{db}} = 0$$

8. Za nosač (***E***=21 *GPa*) i opterećenje kao na slici odrediti:

a. Reakcije oslonaca i nacrati dijagram raspodjele normalnih sila u nosaču;

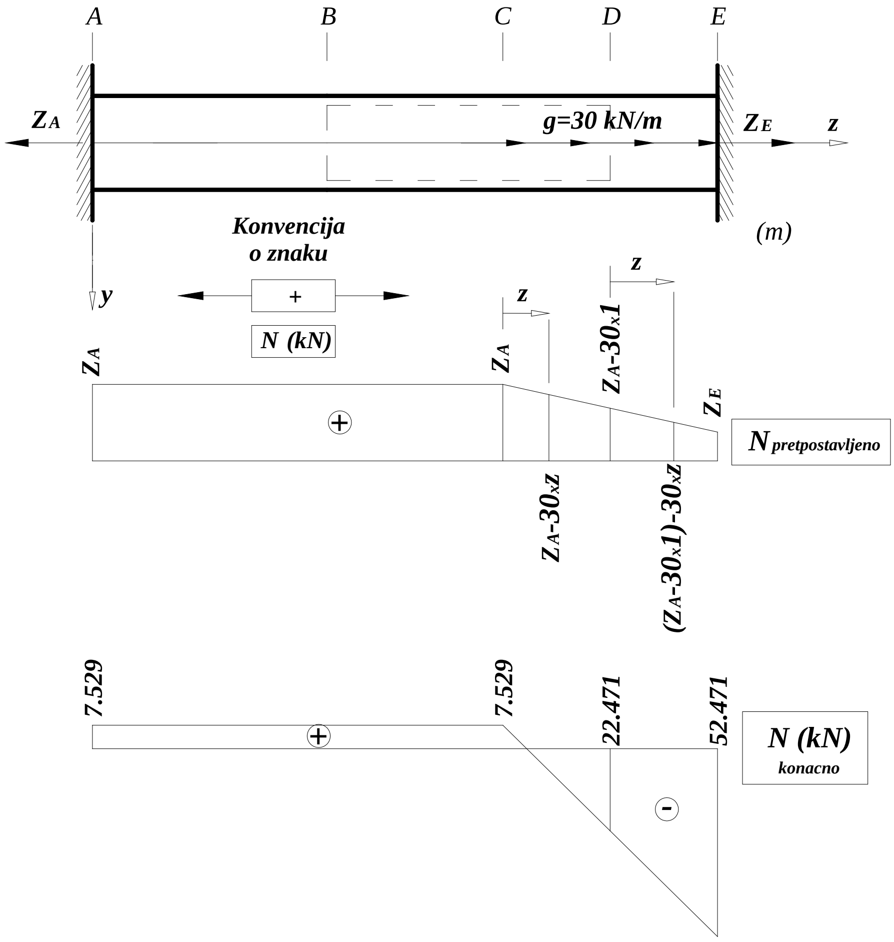
b. Dijagrame normalnih napona u presjecima B_{lijevo} i D_{lijevo};

c. Pomjeranje tačaka u presjecima ***B***, ***C*** i ***D***.



R J E Š E NJ E

a.



$$\Delta l = \int_0^l \frac{N(z)}{EA(z)} dz \quad za\ N = const\ i\ A = const \ \rightarrow \ \Delta l = \frac{N}{EA}\int_0^l dz = \frac{Nl}{EA}$$

Površine poprečnih presjeka:

$A_1 = 20^2 = 400\ cm^2$

$A_2 = 20^2 - 14^2 = 204\ cm^2$

Uslov ravnoteže:

$$\Sigma Z = 0 \;\rightarrow\; g \cdot 2 = Z_A - Z_E \;\rightarrow\; Z_E = Z_A - 30 \cdot 2$$

Deformacijski uslov:

$$\Delta l = 0 \;\rightarrow\; \Delta l_{AB} + \Delta l_{BC} + \Delta l_{CD} + \Delta l_{DE} = 0$$

$$\frac{Z_A \cdot 3}{EA_1} + \frac{Z_A \cdot 2}{EA_2} + \int_0^1 \frac{(Z_A - g \cdot z)}{EA_2} dz + \int_0^1 \frac{[(Z_A - g \cdot 1) - g \cdot z]}{EA_1} dz = 0 \quad \Big| \cdot EA_1$$

$$3Z_A + \frac{A_1}{A_2} 2Z_A + \frac{A_1}{A_2}\left[Z_A \cdot z - g\frac{z^2}{2}\right]\Bigg|_0^1 + \left[(Z_A - g \cdot 1) \cdot z - g\frac{z^2}{2}\right]\Bigg|_0^1 = 0$$

$$3Z_A + 1.961 \cdot 2Z_A + 1.961\left[Z_A \cdot 1 - 30\frac{1^2}{2}\right] + \left[(Z_A - 30 \cdot 1) \cdot 1 - 30\frac{1^2}{2}\right] = 0 \;\rightarrow\; Z_A = 7.529\,kN$$

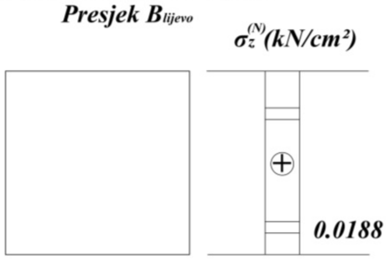
$$Z_E = -52.47\,kN$$

b.

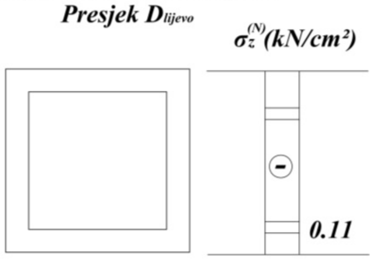
$$\sigma_z^{B_{lijevo}} = \frac{7.529}{400} = 0.0188\,\frac{kN}{cm^2}$$

$$\sigma_z^{D_{lijevo}} = \frac{-22.471}{204} = -0.11\,\frac{kN}{cm^2}$$

Dijagram normalnih napona
Presjek B_{lijevo}



Dijagram normalnih napona
Presjek D_{lijevo}



c.

Pomjeranje presjeka ***B***

$$\Delta l_{AB} = \frac{N_{AB} \cdot l_{AB}}{E \cdot A_{AB}} = \frac{7.529 \cdot 300}{21 \cdot 10^2 \cdot 400} = 0.00269\;cm$$

$$\Delta l_{AC} = \Delta l_{AB} + \Delta l_{BC} = \Delta l_{AB} + \frac{N_{BC} \cdot l_{BC}}{E \cdot A_{BC}} = 0.00269 + \frac{7.529 \cdot 200}{21 \cdot 10^2 \cdot 204} = 0.00269 + 0.00351 = 0.0062\;cm$$

$$\Delta l_{AD} = \Delta l_{AC} + \Delta l_{CD} = \Delta l_{AC} + \int_0^1 \frac{(7.529 - 30 \cdot z)}{EA_2} dz =$$

$$= 0.0062 + \frac{7.529 \cdot 100 - 30 \cdot 10^{-2} \cdot \frac{100^2}{2}}{21 \cdot 10^2 \cdot 204} = 0.0062 - 0.00174 = 0.00446\;cm$$

Kontrola $\Delta l_{AE} = 0$

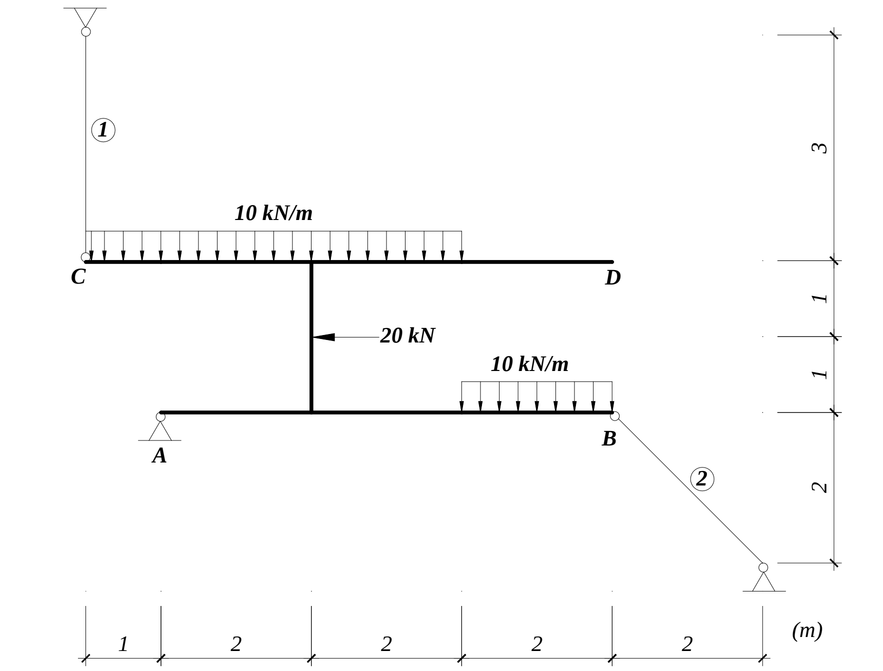
$$\Delta l_{AE} = \Delta l_{AD} + \Delta l_{DE} = \Delta l_{AD} + \int_0^1 \frac{(-22.471 - 30 \cdot z)}{EA_1} dz =$$

$$= 0.00446 + \frac{-22.471 \cdot 100 - 30 \cdot 10^{-2} \cdot \frac{100^2}{2}}{21 \cdot 10^2 \cdot 400} = 0.00446 - 0.00446 = 0\;cm$$

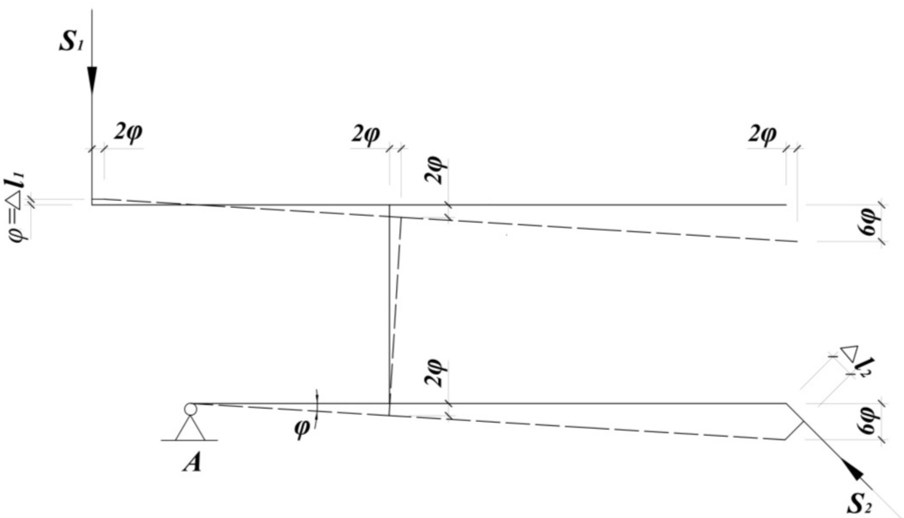
VJEŽBA BR. 10

AKSIJALNO NAPREZANJE II

1. Apsolutno krutu gredu ***ABCD*** u ravnoteži održavaju dva deformabilna štapa. Sračunati sile u štapovima. $\left(A_1 = 0.7A_2;\ E_1 = E_2\right)$



R J E Š E NJ E



Uslov ravnoteže:

$$\Sigma M_A = 0 \quad \rightarrow \quad S_1 \cdot 1 + S_2 \frac{\sqrt{2}}{2} \cdot 6 - 10 \cdot 5 \cdot 1.5 - 10 \cdot 2 \cdot 5 + 20 \cdot 1 = 0$$

$$S_1 + S_2 \cdot 3\sqrt{2} = 155 \qquad (1)$$

Deformacijski uslov:

$$\left.\begin{aligned} \Delta l_1 = \varphi \quad &\rightarrow \quad \varphi = \Delta l_1 \\ \frac{\Delta l_2}{\cos 45^0} = 6\varphi \quad &\rightarrow \quad \varphi = \frac{\Delta l_2}{6\cos 45^0} \end{aligned}\right\} \rightarrow \ \Delta l_1 = \frac{\Delta l_2}{6\cos 45^0} \ \rightarrow \ \frac{S_1 \cdot 3}{E \cdot 0.7A_2} = \frac{S_2 \cdot 2\sqrt{2}}{6\sqrt{2}/2 \cdot E \cdot A_2} \ \rightarrow \ S_1 = 0.156 S_2 \quad (2)$$

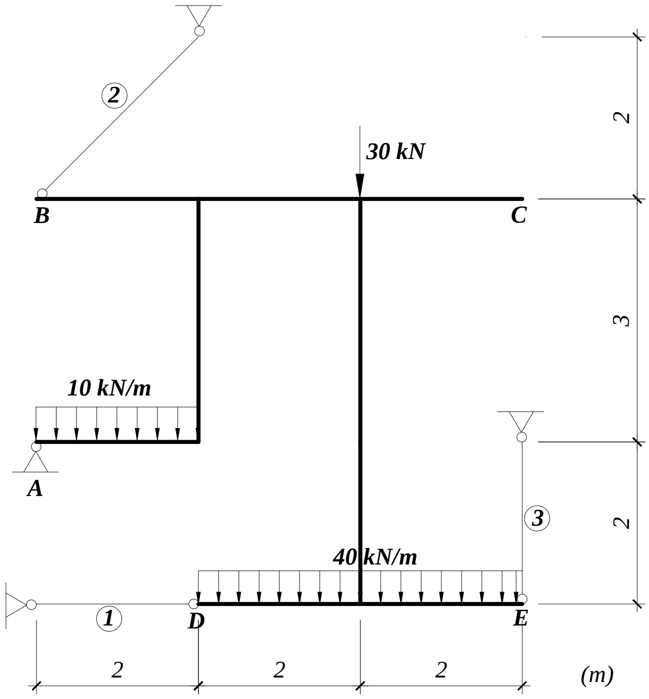
$$(2) \rightarrow (1): \qquad 0.156 S_2 + S_2 \cdot 3\sqrt{2} = 155$$

$$S_2 = 35.24\,kN$$

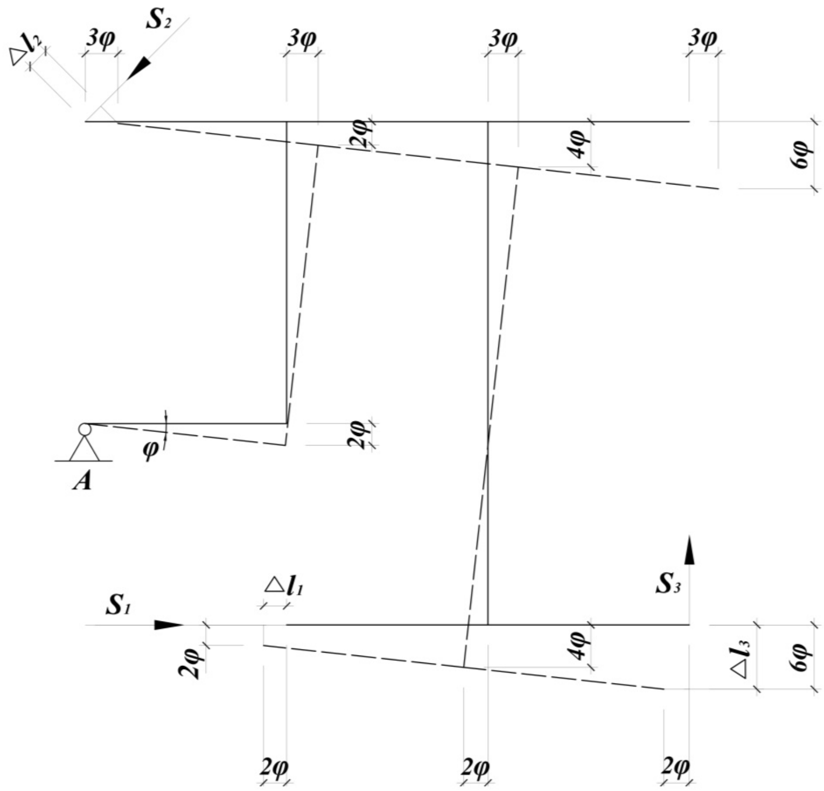
$$S_1 = 5.5\,kN$$

2. Apsolutno krutu gredu ***ABCDE*** u ravnoteži održavaju tri deformabilna štapa. Dimenzionisati štapove ako je:

$A_1 = 0.8A_2 = 0.4A_3;\ E_1 = E_2 = E_3 = E;\ \sigma_{dop} = 10\ \dfrac{kN}{cm^2}$



R J E Š E NJ E



Uslov ravnoteže:

$$\Sigma M_A = 0 \quad \rightarrow \quad S_1 \cdot 2 + S_2 \frac{\sqrt{2}}{2} \cdot 3 + S_3 \cdot 6 = 10 \cdot 2 \cdot 1 + 30 \cdot 4 + 40 \cdot 4 \cdot 4$$

$$2S_1 + \frac{3}{2}\sqrt{2}S_2 + 6S_3 = 780 \qquad (1)$$

Deformacijski uslovi:

$$\left.\begin{array}{lll} \Delta l_1 = 2\varphi & \rightarrow & \varphi = \dfrac{\Delta l_1}{2} \\ \dfrac{\Delta l_2}{\cos 45^0} = 3\varphi & \rightarrow & \varphi = \dfrac{\Delta l_2}{3\cos 45^0} \\ \Delta l_3 = 6\varphi & \rightarrow & \varphi = \dfrac{\Delta l_3}{6} \end{array}\right\} \rightarrow \begin{array}{ll} \dfrac{\Delta l_1}{2} = \dfrac{\Delta l_2}{3\cos 45^0} \rightarrow \dfrac{S_1 \cdot 2}{2 \cdot E \cdot 0.8 A_2} = \dfrac{S_2 \cdot 2\sqrt{2}}{3\sqrt{2}/2 \cdot E \cdot A_2} \rightarrow S_2 = 0.9375 S_1 & (2) \\ \dfrac{\Delta l_1}{2} = \dfrac{\Delta l_3}{6} \rightarrow \dfrac{S_1 \cdot 2}{2 \cdot E \cdot 0.4 A_3} = \dfrac{S_3 \cdot 2}{6 \cdot E \cdot A_3} \rightarrow S_3 = 7.5 S_1 & (3) \end{array}$$

$$(2)\ (3) \rightarrow (1): \quad 2S_1 + \frac{3}{2}\sqrt{2} \cdot 0.9375 S_1 + 6 \cdot 7.5 S_1 = 780$$

$$S_1 = 15.922\, kN$$
$$S_2 = 14.927\, kN$$
$$S_3 = 119.415\, kN$$

Dimenzionisanje:

$\sigma_z = \dfrac{S}{A} \le \sigma_{dop} \quad \rightarrow$ mjerodavan je štap najmanjeg poprečnog presjeka izložen najvećoj aksijalnoj sili

Mjerodavan je štap 3

$$A_{3(potr)} \ge \frac{119.415}{10} = 11.94\ cm^2 \quad Usvaja\ se\ \ A_3 = 12\ cm^2 \rightarrow A_1 = 0.4 \cdot 12 = 4.8 cm^2 \ \ i \ \ A_2 = \frac{0.4}{0.8} \cdot 12 = 6 cm^2$$

Kontrola napona:

$$\sigma_{z,1} = \frac{S_1}{A_1} = \frac{15.922}{4.8} = 3.32 \frac{kN}{cm^2} \le 10 \frac{kN}{cm^2} = \sigma_{dop}$$
$$\sigma_{z,2} = \frac{S_2}{A_2} = \frac{14.927}{6} = 2.49 \frac{kN}{cm^2} \le 10 \frac{kN}{cm^2} = \sigma_{dop}$$
$$\sigma_{z,3} = \frac{S_3}{A_3} = \frac{119.415}{12} = 9.95 \frac{kN}{cm^2} \le 10 \frac{kN}{cm^2} = \sigma_{dop}$$

3. Apsolutno krutu gredu ***ABCD*** u ravnoteži održavaju tri deformabilna štapa. Odrediti:

a. Sile u štapovima;

b. Pomjeranje tačke ***A***.

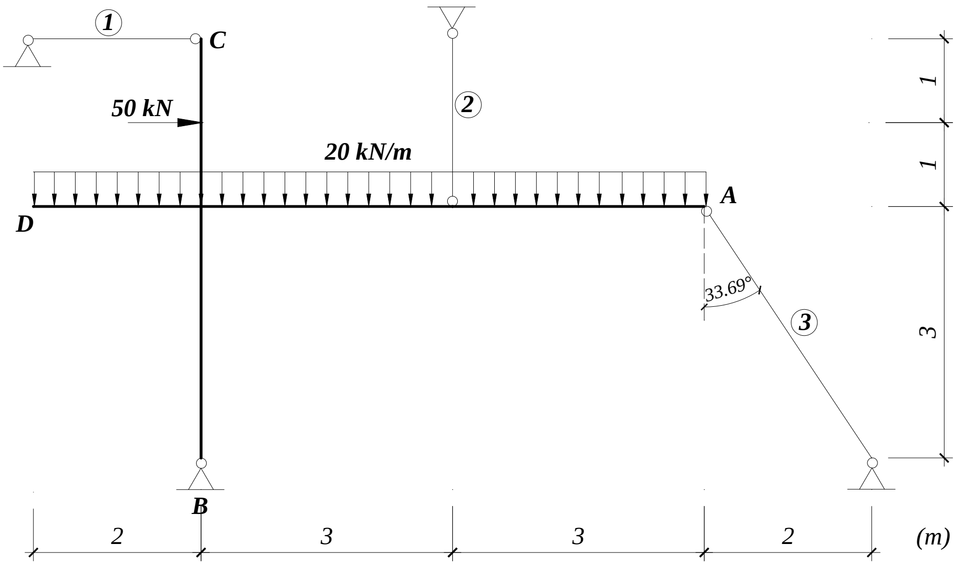
$A_1 = A$

$A_2 = A$

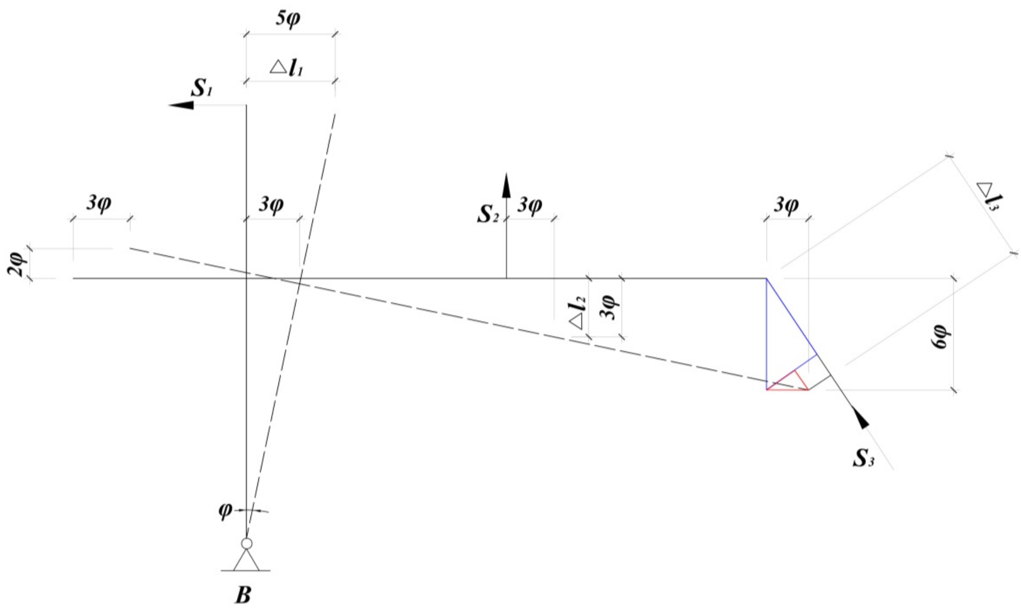
$A_3 = 3A$

$A = 5cm^2$

$E_1 = E_2 = E_3 = E = 2.1 \cdot 10^4 \dfrac{kN}{cm^2}$



R J E Š E NJ E



a. Sile u štap

$$tg\alpha = \frac{2}{3} \quad \alpha = arctg\frac{2}{3} = 33.69^0 \;\rightarrow\; \begin{array}{l} \sin\alpha = 0.5547 \\ \cos\alpha = 0.832 \end{array}$$

Uslov ravnoteže:

$$\Sigma M_B = 0 \;\rightarrow\; S_1 \cdot 5 + S_2 \cdot 3 + S_3 \cos\alpha \cdot 6 + S_3 \sin\alpha \cdot 3 = 20 \cdot 8 \cdot 2 + 50 \cdot 4$$

$$5S_1 + 3S_2 + 6.6561S_3 = 520 \qquad (1)$$

Deformacijski uslovi:

$$\left.\begin{aligned}
&\Delta l_1 = 5\varphi \quad \rightarrow \quad \varphi = \frac{\Delta l_1}{5} \\
&\Delta l_2 = 3\varphi \quad \rightarrow \quad \varphi = \frac{\Delta l_2}{3} \\
&\Delta l_3 = 6.6561\varphi \quad \rightarrow \quad \varphi = \frac{\Delta l_3}{6.6561} \\
&\Delta l_3 = 6\varphi\cos\alpha + 3\varphi\sin\alpha = 6.6561\varphi
\end{aligned}\right\} \rightarrow$$

$$\frac{\Delta l_1}{5} = \frac{\Delta l_2}{3} \rightarrow \frac{S_1 \cdot 2}{5 \cdot E \cdot A} = \frac{S_2 \cdot 2}{3 \cdot E \cdot A} \rightarrow S_2 = \frac{3}{5} S_1 \qquad (2)$$

$$\frac{\Delta l_1}{5} = \frac{\Delta l_3}{6.6561} \rightarrow \frac{S_1 \cdot 2}{5 \cdot E \cdot A} = \frac{S_3 \cdot \sqrt{13}}{6.6561 \cdot E \cdot 3A} \rightarrow S_3 = 2.2152 S_1 \qquad (3)$$

$$(2)\ (3) \rightarrow (1): \quad 5S_1 + 3 \cdot \frac{3}{5} S_1 + 6.6561 \cdot 2.2152 S_1 = 520$$

$$S_1 = 24.14\, kN$$
$$S_2 = 14.48\, kN$$
$$S_3 = 53.47\, kN$$

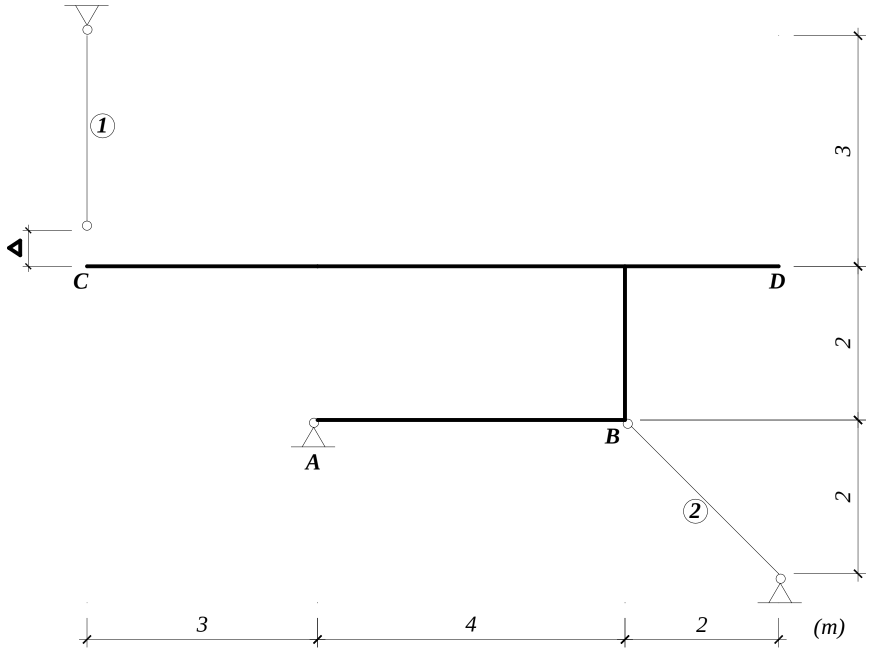
b. Pomjeranje tačke ***A***

$$\left.\begin{aligned}
&\Delta l_1 = \frac{S_1 \cdot l_1}{E \cdot A_1} = \frac{24.14 \cdot 200}{2.1 \cdot 10^4 \cdot 5} = 0.04597\ cm \\
&\Delta l_1 = 5\varphi
\end{aligned}\right\} \rightarrow \varphi = \frac{\Delta l_1}{5} = \frac{0.04597}{5 \cdot 10^2} = 0.009194 \cdot 10^{-2}\ rad$$

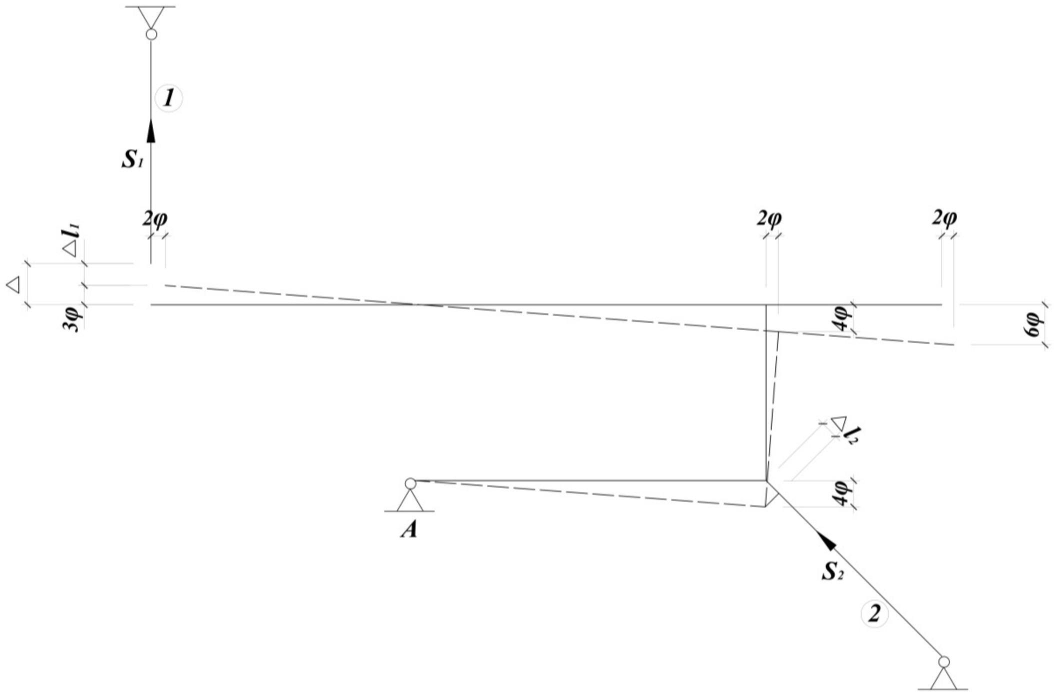
$$\delta_{HOR}{}^A = 3\varphi = 3 \cdot 10^2 \cdot 0.009194 \cdot 10^{-2} = 0.02758\ cm$$
$$\delta_{VER}{}^A = 6\varphi = 6 \cdot 10^2 \cdot 0.009194 \cdot 10^{-2} = 0.05517\ cm$$
$$\delta^A = \sqrt{\left(\delta_{VER}{}^A\right)^2 + \left(\delta_{VER}{}^A\right)^2} = \sqrt{0.02758^2 + 0.05517^2} = 0.06168\ cm$$

$$\Delta l_3 = \frac{S_3 \cdot l_3}{E \cdot A_3} = \frac{53.47 \cdot \sqrt{13} \cdot 100}{2.1 \cdot 10^4 \cdot 3 \cdot 5} = 0.0612\ cm$$

4. Izvršena je prisilna montaža apsolutno krute grede ***ABCD*** pošto je štap broj ***1*** izrađen kraći za $\Delta = 2\,mm$ .Odrediti sile u štapovima ako je poznato: $E_1 = E_2 = E = 210\ GPa;\ A_1 = 3cm^2;\ A_2 = 5cm^2$



R J E Š E NJ E



Uslov ravnoteže:

$\Sigma M_A = 0 \quad \rightarrow \quad S_1 \cdot$

Deformacijski uslov:

$$\left.\begin{aligned} \Delta - \Delta l_1 = 3\varphi \quad &\rightarrow \quad \varphi = \frac{\Delta - \Delta l_1}{3} \\ \frac{\Delta l_2}{\cos 45^0} = 4\varphi \quad &\rightarrow \quad \varphi = \frac{\Delta l_2}{4 \cdot \frac{\sqrt{2}}{2}} \end{aligned}\right\} \rightarrow \frac{\Delta - \Delta l_1}{3} = \frac{\Delta l_2}{4 \cdot \frac{\sqrt{2}}{2}} \rightarrow \quad \Delta - \frac{S_1 \cdot 300}{210 \cdot 10^2 \cdot 3} = \frac{3}{4}\frac{S_2 \cdot 200\sqrt{2}}{\frac{\sqrt{2}}{2} \cdot 210 \cdot 10^2 \cdot 5}$$

$$0.2 - 0.004762 S_1 = 0.0028571 S_2 \qquad (2)$$

$(1) \rightarrow (2):\qquad 0.2 - 0.004762 \cdot \frac{2}{3}\sqrt{2} S_2 = 0.0028571 S_2$

$S_1 = 25.66\,kN$

$S_2 = 27.22\,kN$

VJEŽBA BR. 11

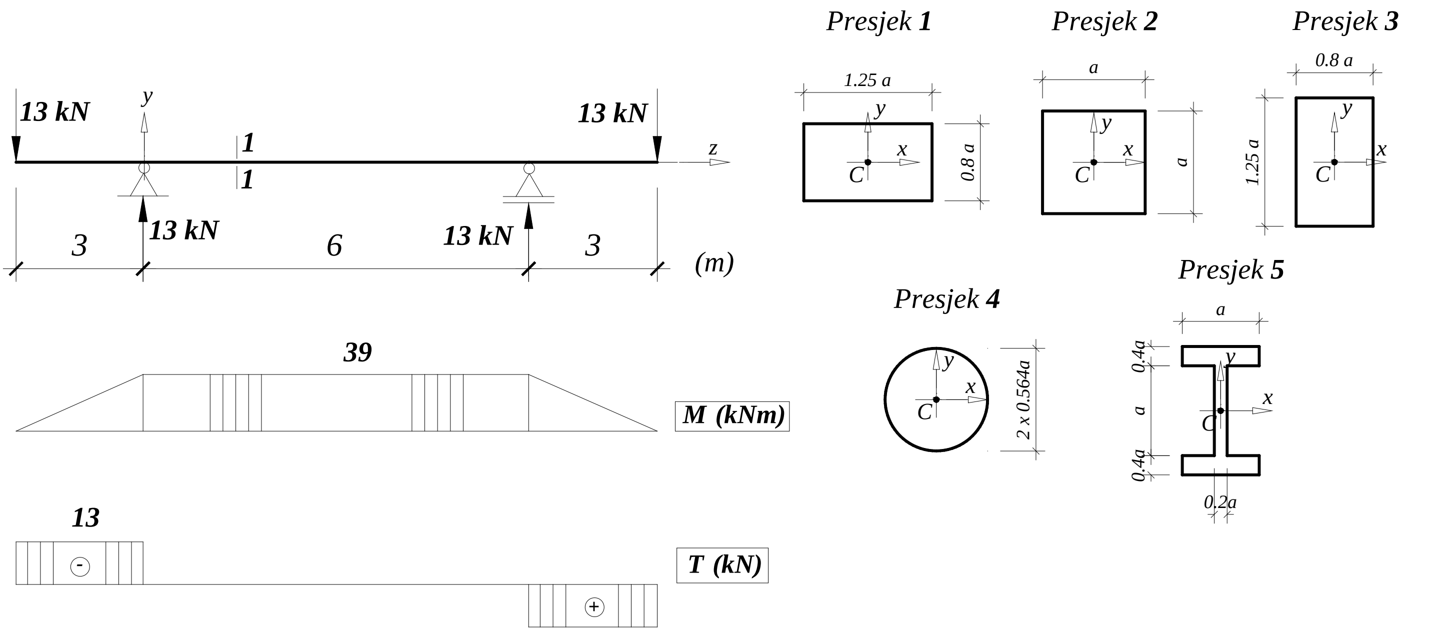
ČISTO PRAVO SAVIJANJE

1.

Za nosač na slici:

a. Od ponuđenih oblika poprečnih presjeka (istih površina) izračunati najpovoljniji i dimenzionisati ga za presjek *1-1* ako je $\sigma_{doz} = 12\,MPa$;

b. Za odabrani i dimenzionisani poprečni presjek nacrtati dijagram normalnih napona u presjeku *1-1*.



R J E Š E NJ E

a. Odabir najpovoljnijeg presjeka i dimenzionisanje:

$\sigma_{z,\max} \le \sigma_{doz}$ - uslov za dimenzionisanje

$\sigma_{z,\max} = \dfrac{M}{I_x} y_{\max} = \dfrac{M}{W_x}$ $\quad \left(W_x = \dfrac{I_x}{y_{\max}} \right)$ - presjek sa najvećim otpornim momentom je najpovoljniji

Pravougaoni poprečni presjek dimenzija ***b*** x ***h*** - $W_x = \dfrac{I_x}{y_{\max}} = \dfrac{\frac{b \cdot h^3}{12}}{\frac{h}{2}} = \dfrac{b \cdot h^2}{6}$

Presjek ***1***: $\quad W_x = \dfrac{b \cdot h^2}{6} = \dfrac{1.25a \cdot (0.8a)^2}{6} = 0.1333a^3$

Presjek ***2***: $\quad W_x = \dfrac{b \cdot h^2}{6} = \dfrac{a \cdot (a)^2}{6} = 0.1666a^3$

Presjek ***3***: $\quad W_x = \dfrac{b \cdot h^2}{6} = \dfrac{0.8a \cdot (1.25a)^2}{6} = 0.2083a^3$

Presjek ***4***: $\quad W_x = \dfrac{I_x}{y_{\max}} = \dfrac{\frac{\pi \cdot R^4}{4}}{R} = \dfrac{\pi \cdot R^3}{4} = \dfrac{\pi \cdot (0.564a)^3}{4} = 0.141a^3$

Presjek ***5***: $I_x = 2 \cdot \frac{a \cdot (0.4a)^3}{12} + 2 \cdot 0.4a \cdot a \cdot (0.7a)^2 + \frac{0.2a \cdot a^3}{12} = 0.4193a^4$

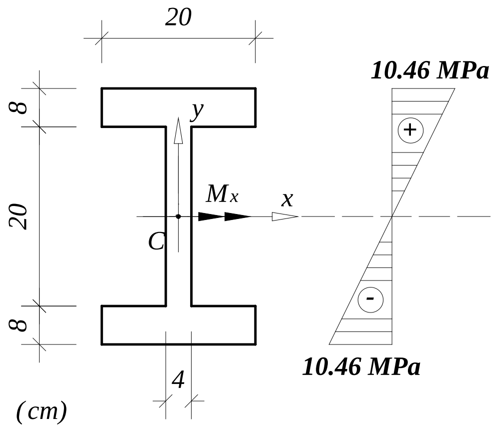
$W_x = \frac{I_x}{y_{max}} = \frac{0.4193a^4}{0.9a} = 0.466a^3$ - Presjek ***5*** je najpovoljniji

$$\sigma_{z,max} = \frac{M}{W_x} \le \sigma_{doz} \quad \rightarrow \quad W_x \ge \frac{M}{\sigma_{doz}} \quad \rightarrow \quad 0.466a^3 \ge \frac{39 \cdot 10^2}{1.2} \quad \rightarrow \quad a \ge 19.1\, cm$$

Usvaja se: $\boxed{a = 20\,cm}$ $\rightarrow$ $W_x = 0.466 \cdot 20^3 = 3728\ cm^3$

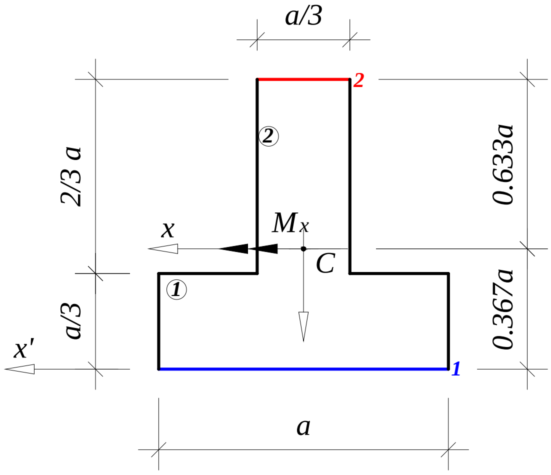
b. Kontrola, odnosno ekstremni normalni naponi:

$$\sigma_{z,\substack{max \\ min}} = \pm \frac{M}{W_x} = \pm \frac{39 \cdot 10^2}{3728} = \pm 1.046\ \frac{kN}{cm^2} \le \sigma_{doz}$$



2.

Greda poprečnog presjeka kao na slici opterećena je na savijanje spregovima M_x=15 *kNm*. Ako je dozvoljeni napon na zatezanje $\sigma_{doz}^{\boxplus} = 20\ MPa$, a na pritisak $\sigma_{doz}^{\boxminus} = 80\ MPa$, izvršiti dimenzionisanje poprečnog presjeka grede i nacrtati dijagram normalnih napona.



RJEŠENJE

Geometrijske karakteristike poprečnog presjeka:

$$C_1\left(0; -\frac{a}{6}\right) \qquad A_1 = \frac{a^2}{3}$$

$$C_2\left(0; -\frac{2}{3}a\right) \qquad A_2 = \frac{2}{9}a^2$$

$$y_C = -\frac{\frac{a}{6}\cdot\frac{a^2}{3} + \frac{2}{3}a\cdot\frac{2}{9}a^2}{\frac{a^2}{3} + \frac{2}{9}a^2} = -0.367a$$

$$I_x = \frac{a\cdot\left(\frac{a}{3}\right)^3}{12} + \frac{a^2}{3}\cdot(0.2a)^2 + \frac{\frac{a}{3}\cdot\left(\frac{2}{3}a\right)^3}{12} + \frac{2}{9}a^2\cdot(-0.3a)^2 = 0.04465a^4$$

Dimenzionisanje:

1 – Zategnuto vlakno ***max***

$$\sigma_{z,\max} = \frac{M}{W_{x,1}} \le \sigma_{doz}^{\boxplus} \quad\rightarrow\quad W_{x,1} \ge \frac{M}{\sigma_{doz}^{\boxplus}}$$

$$W_{x,1} = \frac{I_x}{y_1} = \frac{0.04465a^4}{0.367a} = 0.12177a^3 \ge \frac{15\cdot10^2}{2} = 750\,cm^3 \quad\rightarrow\quad a \ge 18.33\ cm$$

2 – Pritisnuto vlakno ***min***

$$\left|\sigma_{z,\min}\right| = \frac{M}{W_{x,2}} \le \sigma_{doz}^{\boxminus} \quad\rightarrow\quad W_{x,2} \ge \frac{M}{\sigma_{doz}^{\boxminus}}$$

$$W_{x,2} = \frac{I_x}{y_2} = \frac{0.04465a^4}{0.633a} = 0.0705a^3 \ge \frac{15\cdot10^2}{8} = 187.5\,cm^3 \quad\rightarrow\quad a \ge 13.85\ cm$$

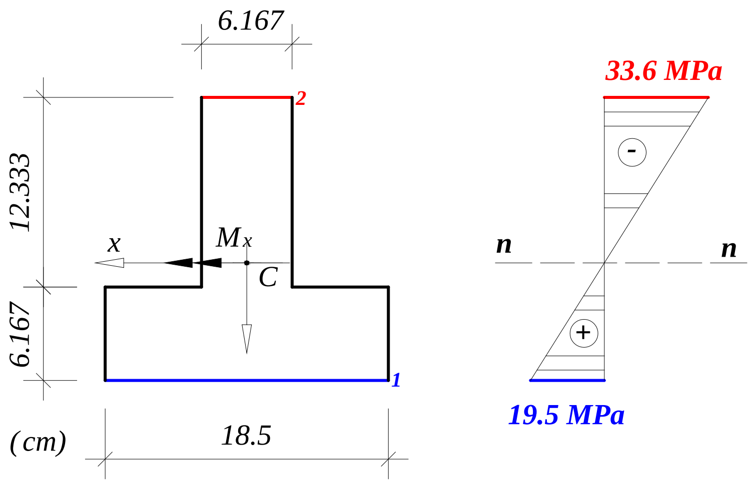
Usvaja se: $\boxed{a = 18.5\,cm}$ $\rightarrow$

$$W_{x,1} = 0.12177 \cdot 18.5^3 = 771\,cm^3$$
$$W_{x,2} = 0.0705 \cdot 18.5^3 = 446.4\,cm^3$$

Kontrola, odnosno ekstremni normalni naponi:

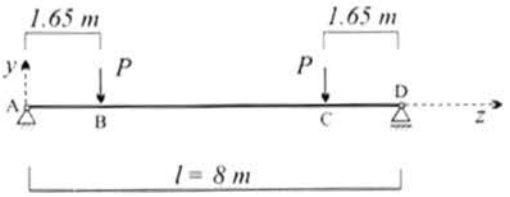
$$\sigma_{z,\max} = \frac{15 \cdot 10^2}{771} = 1.95\,\frac{kN}{cm^2} \le \sigma_{doz}^{\boxed{+}} = 2\,\frac{kN}{cm^2}$$

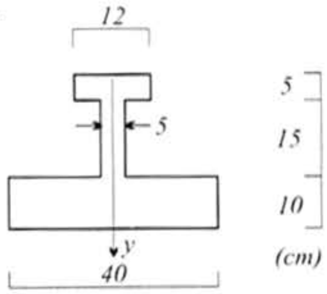
$$\left|\sigma_{z,\min}\right| = \frac{15 \cdot 10^2}{446.4} = 3.36\,\frac{kN}{cm^2} \le \sigma_{doz}^{\boxed{-}} = 8\,\frac{kN}{cm^2}$$



3.

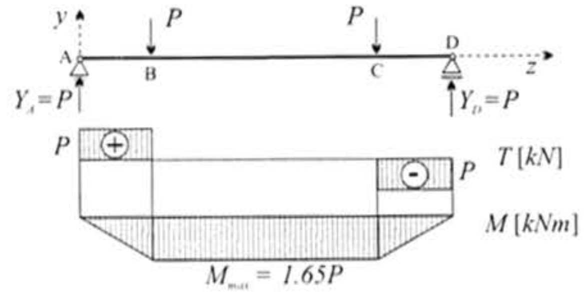
Data je prosta greda opterećena prema slici. Opterećenje leži u ravni yz.
a/ Odrediti maksimalni intenzitet sila P koje može da nosi greda zadatog poprečnog preseka, ako je σ_z = 120 MPa.
b/ Za usvojene vrednosti izvršiti proveru napona σ_z i nacrtati dijagram.





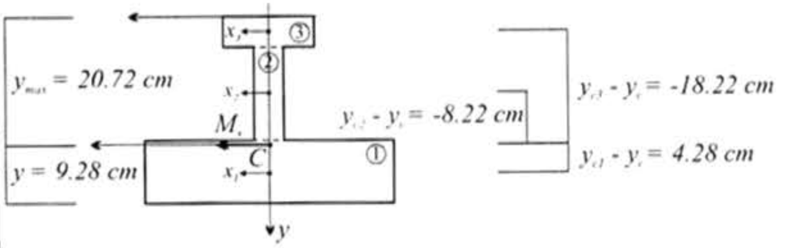
R J E Š E NJ E

a/ Sa dijagrama presečnih sila vidi se da je na delu BC nosač izložen čistom pravom savijanju oko ose x.



Određivanje težišta preseka
Zbog simetrije preseka težište se nalazi na osi simetrije (osi y), koja je ujedno i glavna centralna osa inercije. Znači, treba odrediti samo položaj ose x.

$$y_c = \frac{y_{c1}A_1 + y_{c2}A_2 + y_{c3}A_3}{A_1 + A_2 + A_3} = \frac{25 \cdot 400 + 12.5 \cdot 75 + 2.5 \cdot 60}{400 + 75 + 60} = 20.72 \text{ cm}$$



Geometrijske karakteristike preseka

$$I_x = \frac{40 \cdot 10^3}{12} + 400 \cdot 4.28^2 + \frac{5 \cdot 15^3}{12} + 75 \cdot (-8.22)^2 + \frac{12 \cdot 5^3}{12} + 60 \cdot (-18.22)^2 = 37177.67 \text{ cm}^2$$

$$W_x = \frac{I_x}{y_{max}} = \frac{37177.67}{20.72} = 1794.29 \text{ cm}^3$$

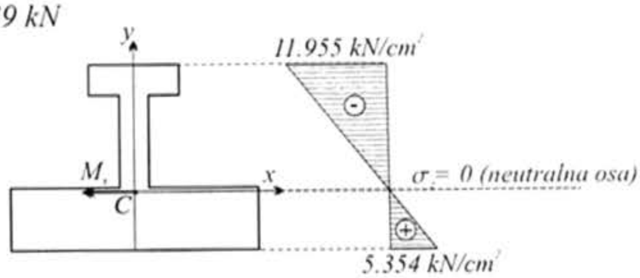
Iz uslova za dimenzionisanje određujemo moment nosivosti.

$$\sigma_{zmax} = \frac{M_x}{I_x} y_{max} = \frac{M_x}{W_x} \leq \sigma_{doz} \quad \Rightarrow \quad M_x = W_x \cdot \sigma_{doz} = 1794.29 \cdot 12 = 21531.5 \text{ kNcm} = 215.315 \text{ kNm}$$

Maksimalna vrednost sile P

$$M_{max} = 1.65 \cdot P \quad \Rightarrow \quad P = \frac{M_{max}}{1.65} = \frac{215.315}{1.65} = 130.49 \text{ kN}$$

usvojeno: P = 130 kN



b/ Stvarni uticaji u nosaču su:

$$M_{max} = 1.65 \cdot P_{usv} = 1.65 \cdot 130 = 214.5 \text{ kNm}$$

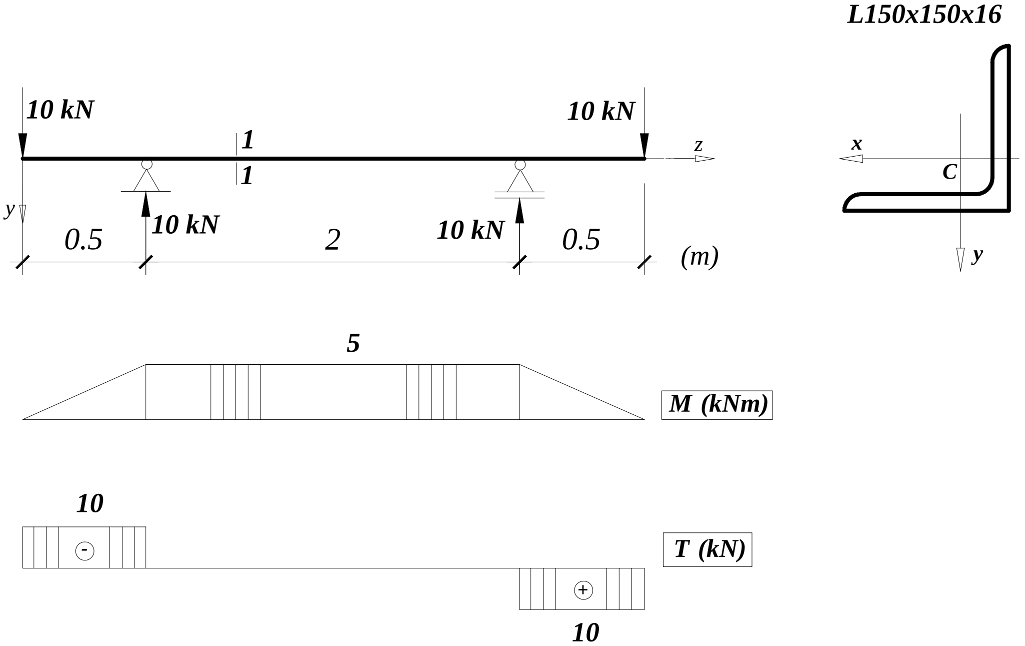
$$\sigma_{zmax} = \frac{M_x}{I_x} y_{max} = \frac{M_x}{W_x} = \frac{214.5 \cdot 10^2}{1794.29}(-20.72) = -11.955 \text{ kN/cm}^2 = -119.55 \text{ MPa}$$

$$\sigma_{z(y=9.28)} = \frac{M_x}{I_x} y = \frac{214.5 \cdot 10^2}{37177.67} 9.28 = 5.354 \text{ kN/cm}^2 = 53.54 \text{ MPa}$$

VJEŽBA BR. 12

ČISTO KOSO SAVIJANJE

1. Poprečni presjek grede sa prepustima je ugaonik ***L150x150x16***. Nacrtati dijagram ekstremnih normalnih napona u presjeku ***1-1*** grede.



R J E Š E NJ E

Geometrijske karakteristike poprečnog presjeka:

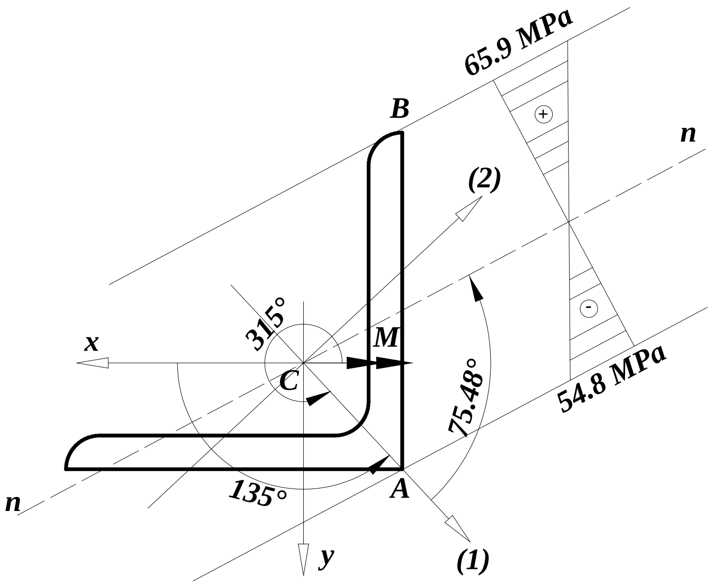
L150x150x16

$e = 4.29\,cm$

$A = 45.7\,cm^2$

$I_1 = 1510\,cm^4$

$I_2 = 391\,cm^4$



Ekstremni normalni naponi:

$\beta = 315^0$

$$tg\varphi = -\frac{I_1}{I_2} tg\beta = -\frac{1510}{391} tg315^0 = 3.862$$

$\varphi = arctg3.862 = 75.48^0$

Tačke u kojima se javljaju ekstremne vrijednosti normalnih napona u sistemu osa ***xy:***

$A(x_A;\, y_A) = (-4.29;\, 4.29)\,cm$

$B(x_B;\, y_B) = (-4.29;\, -10.71)\,cm$

Tačke u kojima se javljaju ekstremne vrijednosti normalnih napona u sistemu glavnih osa ***(1)(2):***

$(1)_A = x_A \cos\alpha + y_A \sin\alpha = (-4.29)\cos135^0 + 4.29\sin135^0 = 6.067\ cm$

$(2)_A = -x_A \sin\alpha + y_A \cos\alpha = 4.29\sin135^0 + 4.29\cos135^0 = 0\ cm$

$$(1)_B = x_B \cos\alpha + y_B \sin\alpha = (-4.29)\cos 135^0 - 10.71 \sin 135^0 = -4.54\ cm$$

$$(2)_B = -x_B \sin\alpha + y_B \cos\alpha = 4.29 \sin 135^0 - 10.71 \cos 135^0 = 10.607\ cm$$

$$A((1)_A;\ (2)_A) = (6.067;\ 0)\,cm$$

$$B((1)_B;\ (2)_B) = (-4.54;\ 10.607)\,cm$$

$$\sigma_z^{\ A,B} = M\left(\frac{\cos\beta}{I_1}\cdot(2)_{A,B} + \frac{\sin\beta}{I_2}\cdot(1)_{A,B}\right)$$

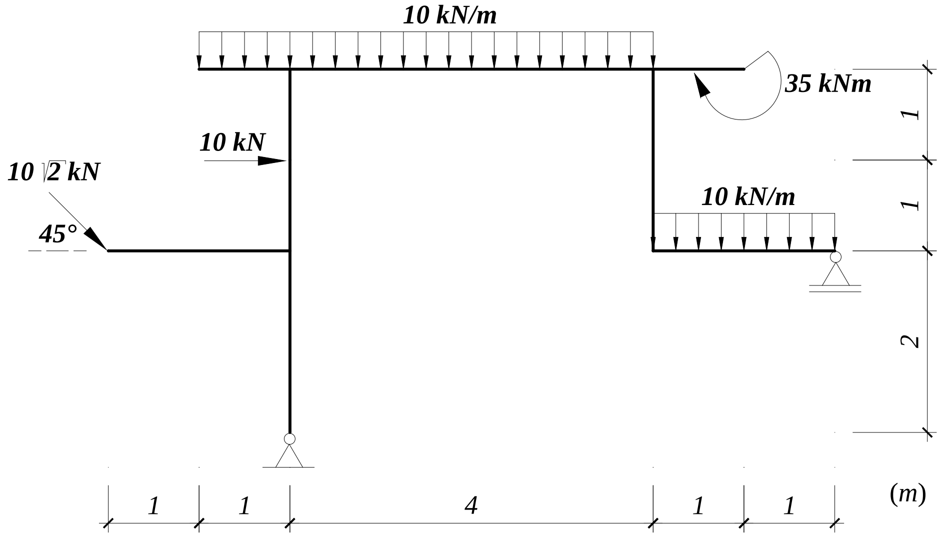
$$\sigma_z^{\ A} = 5\cdot 10^2\left(\frac{\cos 315^0}{1510}\cdot 0 + \frac{\sin 315^0}{391}\cdot 6.067\right) = -5.48\ \frac{kN}{cm^2}$$

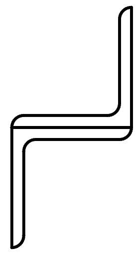
$$\sigma_z^{\ B} = 5\cdot 10^2\left(\frac{\cos 315^0}{1510}\cdot 10.607 + \frac{\sin 315^0}{391}\cdot(-4.54)\right) = 6.59\ \frac{kN}{cm^2}$$

2. Za nosač na slici:

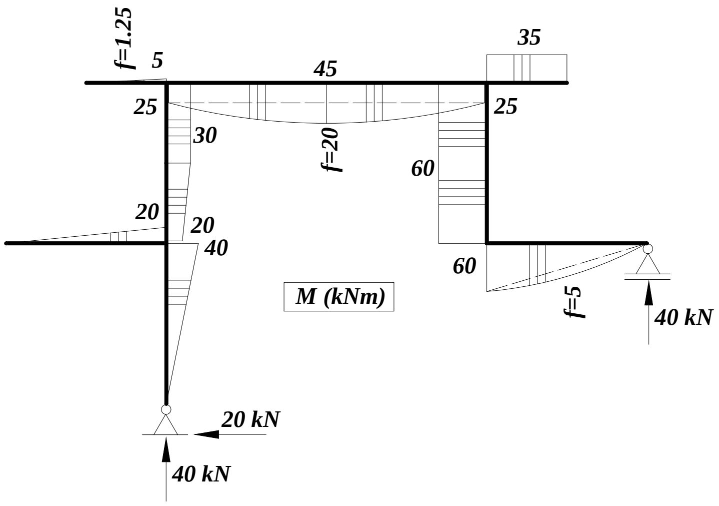
a. Nacrtati dijagrame presječnih sila (*M*, *T*, *N*);

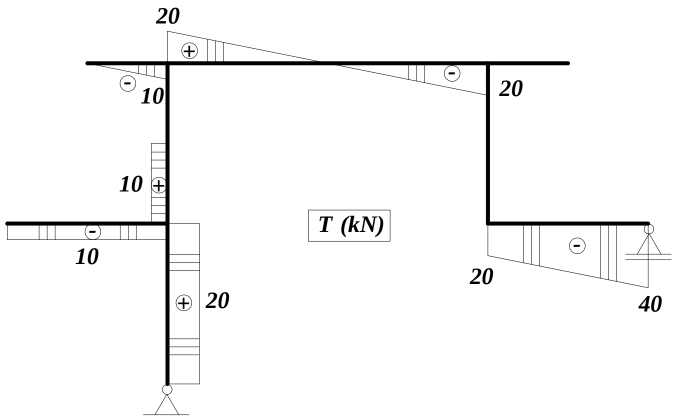
b. Na gornjoj rigli, na mjestu maksimalnog momenta, nacrtati dijagram ekstremnih vrijednosti normalnog napona.

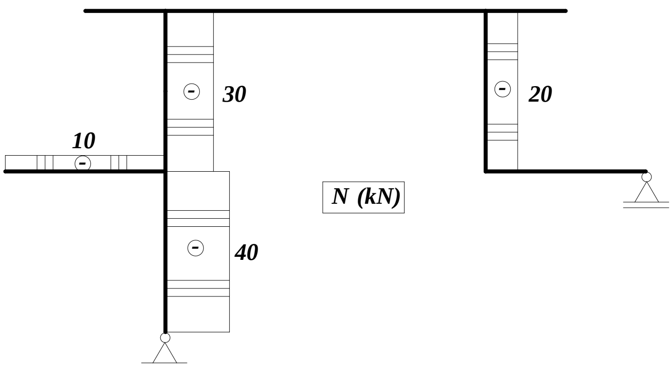




R J E Š E NJ E







Geometrijske karakteristike poprečnog presjeka:

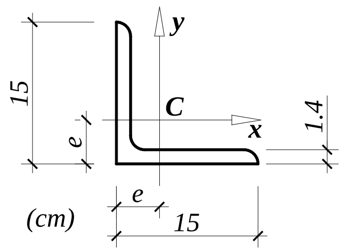
L150x150x14

$e = 4.21cm$

$A = 40.3cm^2$

$I_x = I_y = 845cm^4$

$I_{xy} = -495cm^4$



$$I_\xi = 2\left[845 + 40.3 \cdot 4.21^2\right] = 3118.6\ cm^4$$

$$I_\eta = 2\left[845 + 40.3 \cdot (7.5 - 4.21)^2\right] = 2562.4\ cm^4$$

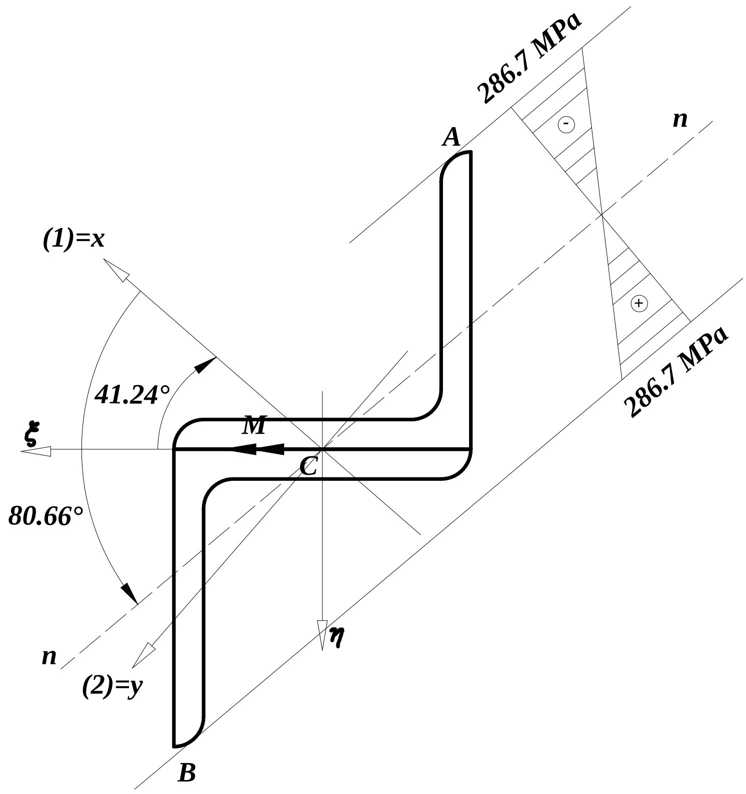
$$I_{\xi\eta} = 2\left[495 + 40.3 \cdot 4.21 \cdot 3.29\right] = 2106.4\ cm^4$$

$$I_{1/2} = \ldots = 2840.5 \pm 2124.7$$

$$I_1 = 4965.2\ cm^4$$

$$I_2 = 715.8\ cm^4$$

$$tg\,2\alpha = \frac{-2 \cdot 2106.4}{3118.6 - 2562.4} = -7.575 \quad \rightarrow \quad \alpha = -41.24^0$$



Ekstremni normalni naponi:

$$\beta = -41.24^0$$

$$tg\varphi = -\frac{I_1}{I_2} tg\beta = -\frac{4968.2}{715.8} tg\left(-41.24^0\right) = 6.081$$

$$\varphi = arctg\,6.081 = 80.66^0$$

Tačke u kojima se javljaju ekstremne vrijednosti normalnih napona u sistemu osa ***ξη:***

$$A(\xi_A;\, \eta_A) = (-7.5;\, -15)\,cm$$

$$B(\xi_B;\, \eta_B) = (7.5;\, 15)\,cm$$

Tačke u kojima se javljaju ekstremne vrijednosti normalnih napona u sistemu glavnih osa ***xy:***

$$x_A = \xi_A \cos\alpha + \eta_A \sin\alpha = (-7.5)\cos\left(-41.24^0\right) + (-15)\sin\left(-41.24^0\right) = 4.25\ cm$$

$$y_A = -\xi_A \sin\alpha + \eta_A \cos\alpha = 7.5\sin\left(-41.24^0\right) + (-15)\cos\left(-41.24^0\right) = -16.22\ cm$$

$$x_B = \xi_B \cos\alpha + \eta_B \sin\alpha = 7.5\cos\left(-41.24^0\right) + 15\sin\left(-41.24^0\right) = -4.25\ cm$$

$$y_B = -\xi_B \sin\alpha + \eta_B \cos\alpha = -7.5\sin\left(-41.24^0\right) + 15\cos\left(-41.24^0\right) = 16.22\ cm$$

$$A(x_A;\, y_A) = (4.25;\, -16.22)\,cm$$

$$B(x_B;\, y_B) = (-4.25;\, 16.22)\,cm$$

$$\sigma_z^{A,B} = M\left(\frac{\cos\beta}{I_1} \cdot y_{A,B} + \frac{\sin\beta}{I_2} \cdot x_{A,B}\right)$$

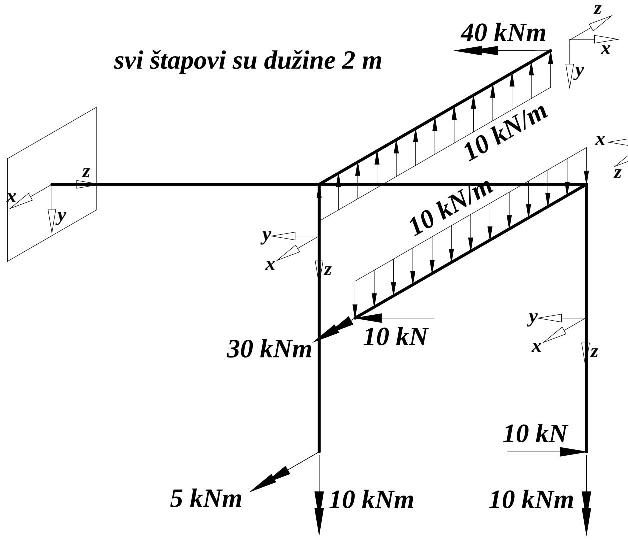
$$\sigma_z^{A} = 45 \cdot 10^2 \left(\frac{\cos\left(-41.25^0\right)}{4965.2} \cdot (-16.22) + \frac{\sin\left(-41.25^0\right)}{715.8} \cdot 4.25\right) = -28.666\ \frac{kN}{cm^2}$$

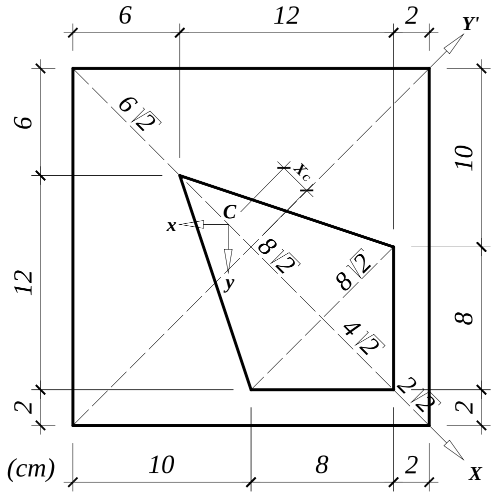
$$\sigma_z^{B} = 45 \cdot 10^2 \left(\frac{\cos\left(-41.25^0\right)}{4965.2} \cdot 16.22 + \frac{\sin\left(-41.25^0\right)}{715.8} \cdot (-4.25)\right) = 28.666\ \frac{kN}{cm^2}$$

3. Za nosač na slici:

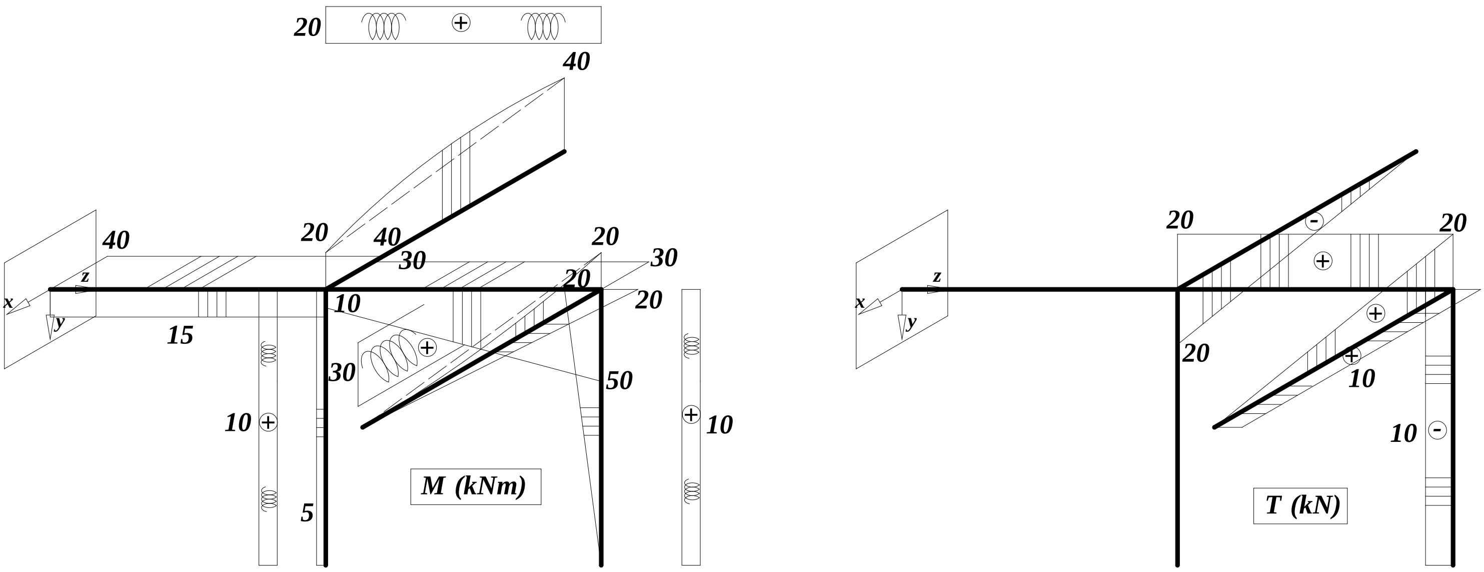
a. Nacrtati dijagrame presječnih sila (*M*, *T*, *N*);

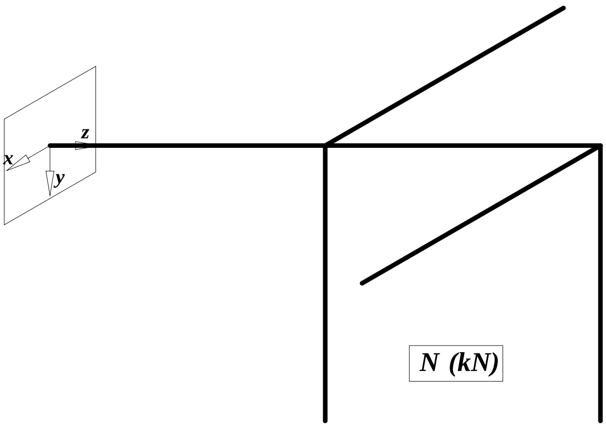
b. Za presjek u uklještenju nacrtati dijagram ekstremnih vrijednosti normalnog napona.





R J E Š E NJ E





Geometrijske karakteristike poprečnog presjeka:

$$X_C = \frac{-1.333\sqrt{2}\cdot 64 - 5.333\sqrt{2}\cdot 32}{304} = -1.191\,cm$$

$$I_X = \frac{20^4}{12} - \frac{1}{48}\left[\left(8\sqrt{2}\right)^4 + 4\sqrt{2}\cdot\left(8\sqrt{2}\right)^3\right] = 12821\ cm^4 \equiv I_1$$

$$I_Y = \frac{20^4}{12} + 400\cdot 1.191^2 - \frac{1}{36}\left(8\sqrt{2}\right)^4 - 64\cdot 3.077^2 - \frac{1}{36}8\sqrt{2}\cdot\left(4\sqrt{2}\right)^3 - 32\cdot 8.733^2 = 10342\ cm^4 \equiv I_2$$

$$I_{XY} = 0$$

Ekstremni normalni naponi:

$$M = \sqrt{15^2 + 40^2} = 42.72\ kNm$$

$$\beta = 45 + arctg\frac{15}{40} = 65.556^0$$

$$tg\varphi = -\frac{I_1}{I_2}tg\beta = -\frac{12821}{10342}tg65.556^0 = -2.7279$$

$$\varphi = arctg\left(-2.7279\right) = -69.864^0$$

Tačke u kojima se javljaju ekstremne vrijednosti normalnih napona u sistemu glavnih osa ***XY***:

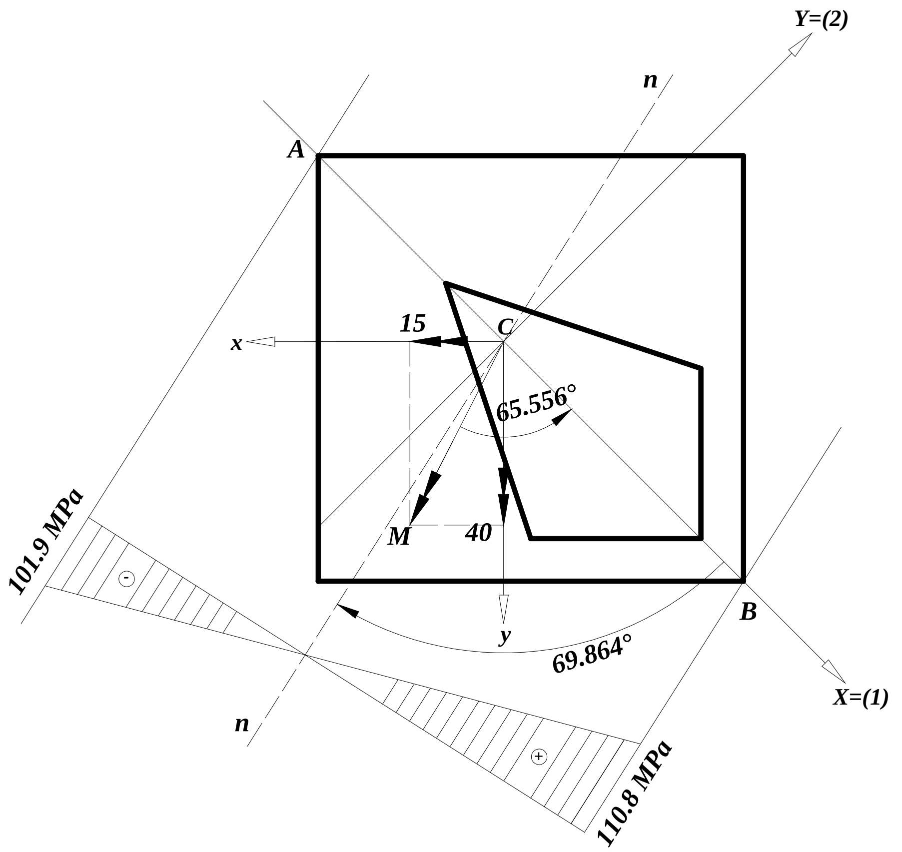
$$A(X_A; Y_A) = (-12.951;\ 0)\,cm$$

$$B(X_B; Y_B) = (15.333;\ 0)\,cm$$

$$\sigma_z^{A,B} = M\left(\frac{\cos\beta}{I_1}\cdot Y_{A,B} + \frac{\sin\beta}{I_2}\cdot X_{A,B}\right)$$

$$\sigma_z^{A} = 42.72\cdot 10^2\left(\frac{\cos 65.556^0}{12821}\cdot 0 + \frac{\sin 65.556^0}{10342}\cdot\left(-12.951\right)\right) = -10.19\ \frac{kN}{cm^2}$$

$$\sigma_z^{B} = 42.72\cdot 10^2\left(\frac{\cos 65.556^0}{12821}\cdot 0 + \frac{\sin 65.556^0}{10342}\cdot 15.333\right) = 11.08\ \frac{kN}{cm^2}$$



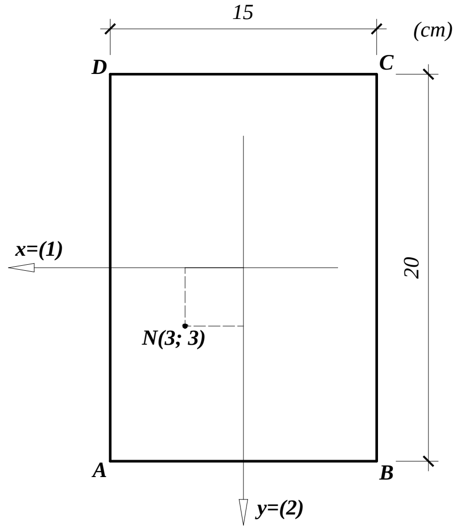
VJEŽBA BR. 13

EKSCENTRIČNO NAPREZANJE

1. Drveni stub pravougaonog poprečnog presjeka $15x20cm$ opterećen je ekscentričnom silom pritiska od $90kN$ koja djeluje u tački ***N***(3;3). Odrediti položaj neutralne linije, dijagram ekstremnih normalnih napona, kao i dijagram normalnih napona duž stranice ***BC*** pravougaonika.

R J E Š E NJ E

N(3;3) $\rightarrow$ a=3cm, b=3cm



Geometrijske karakteristike poprečnog presjeka:

$$A = 15 \cdot 20 = 300\,cm^2$$

$$I_x = \frac{15 \cdot 20^3}{12} = 10\,000\,cm^4$$

$$I_y = \frac{15^3 \cdot 20}{12} = 5625\,cm^4$$

$$i_x^{\,2} = \frac{I_x}{A} = \frac{10\,000}{300} = 33.333\,cm^2$$

$$i_y^{\,2} = \frac{I_y}{A} = \frac{5625}{300} = 18.75\,cm^2$$

Neutralna linija i normalni naponi:

$$p = -\frac{i_y^{\,2}}{a} = -\frac{18.75}{3} = -6.25\ cm$$

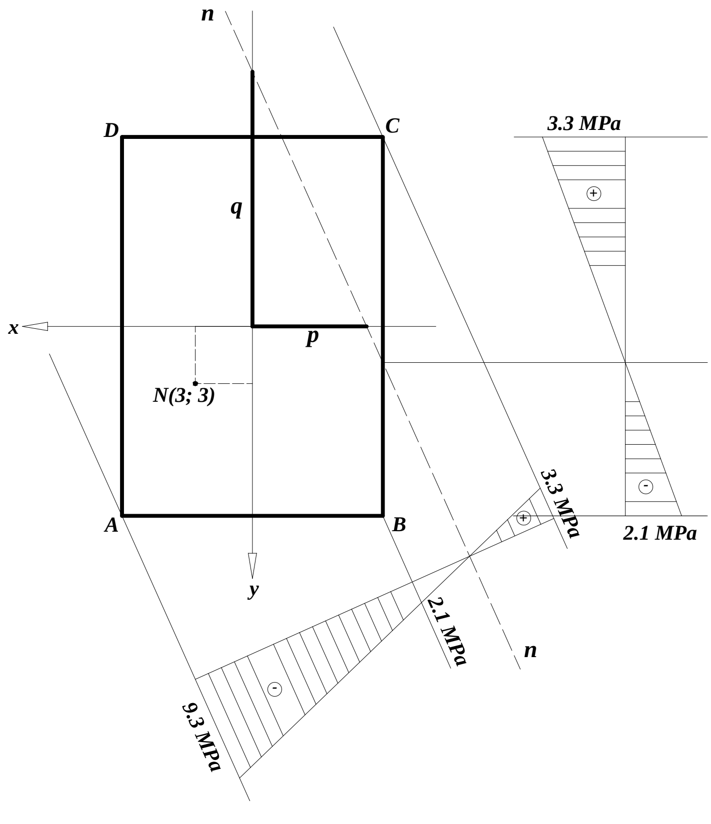
$$q = -\frac{i_x^{\,2}}{b} = -\frac{33.333}{3} = -11.111\ cm$$

$$\sigma_z = \pm\frac{P}{A}\left(1 + \frac{b}{i_x^{\,2}} \cdot y + \frac{a}{i_y^{\,2}} \cdot x\right)$$

$$\sigma_z^{\,A} = -\frac{90}{300}\left(1 + \frac{3}{33.333} \cdot 10 + \frac{3}{18.75} \cdot 7.5\right) = -0.93\frac{kN}{cm^2}$$

$$\sigma_z^{\,C} = -\frac{90}{300}\left(1 + \frac{3}{33.333} \cdot (-10) + \frac{3}{18.75} \cdot (-7.5)\right) = 0.33\frac{kN}{cm^2}$$

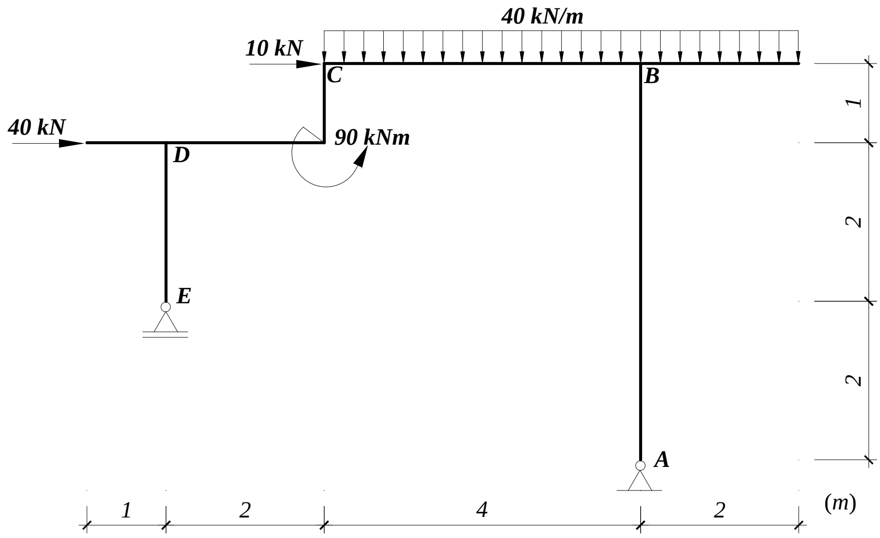
$$\sigma_z^{\,B} = -\frac{90}{300}\left(1 + \frac{3}{33.333} \cdot (-7.5) + \frac{3}{18.75} \cdot 10\right) = -0.21\frac{kN}{cm^2}$$

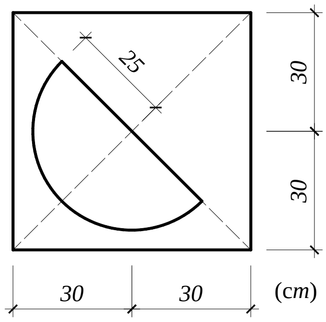


2. Za nosač na slici:

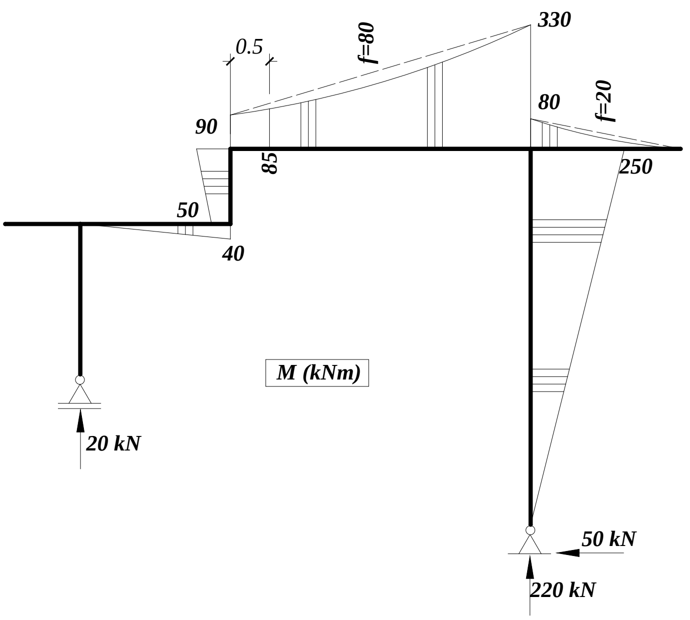
a. Nacrtati dijagrame presječnih sila (*M*, *T*, *N*);

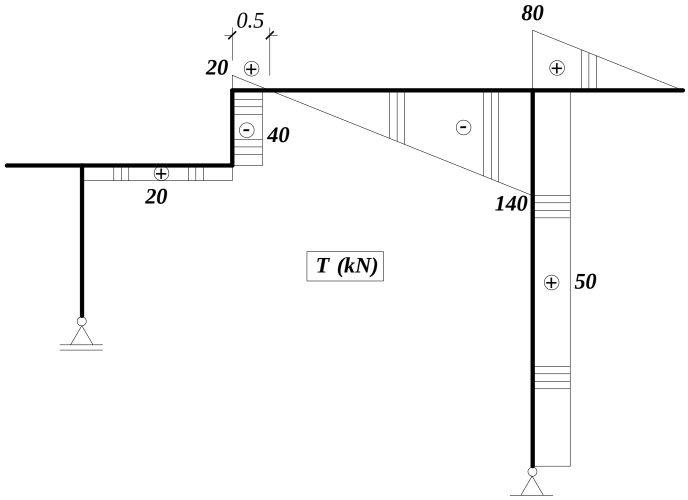
b. Na gornjoj rigli (dio *BC*), na mjestu gdje je transverzalna sila jednaka nuli, nacrtati dijagram ekstremnih vrijednosti normalnog napona.

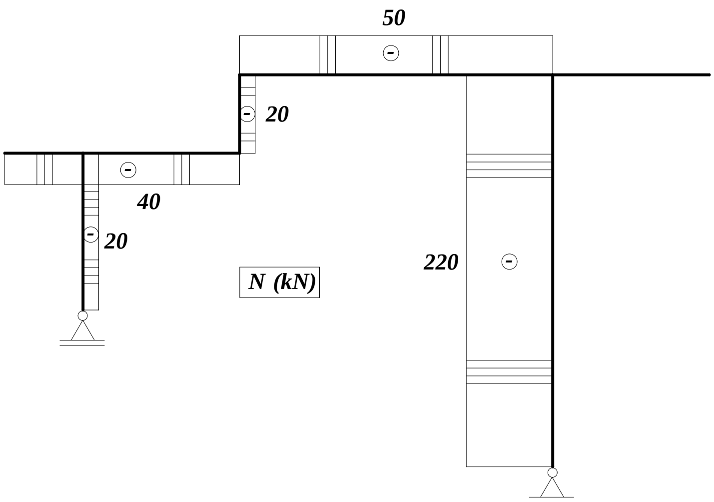




R J E Š E NJ E







Geometrijske karakteristike poprečnog presjeka:

$$C_1(0;0) \quad A_1 = 3600\ cm^2$$

$$C_2\left(-\frac{4\cdot 25}{3\pi};0\right) = (-10.61;0) \quad A_1 = 981.742\ cm^2$$

$$x_C = \frac{-(-10.61)\cdot 981.748}{2618.252} = 3.978\ cm$$

$$I_x = \frac{1}{12}60^4 - \frac{25^4\pi}{8} = 926\,602\ cm^4 \equiv I_1$$

$$I_y = \frac{1}{12}60^4 + 60^2\cdot 3.978^2 - 0.10976\cdot 25^4 - 981.748\cdot 14.588^2 = 885\,168\ cm^4 \equiv I_2$$

$$i_1^{\,2} = i_x^{\,2} = \frac{926\,602}{2618.252} = 353.9\,cm^2$$

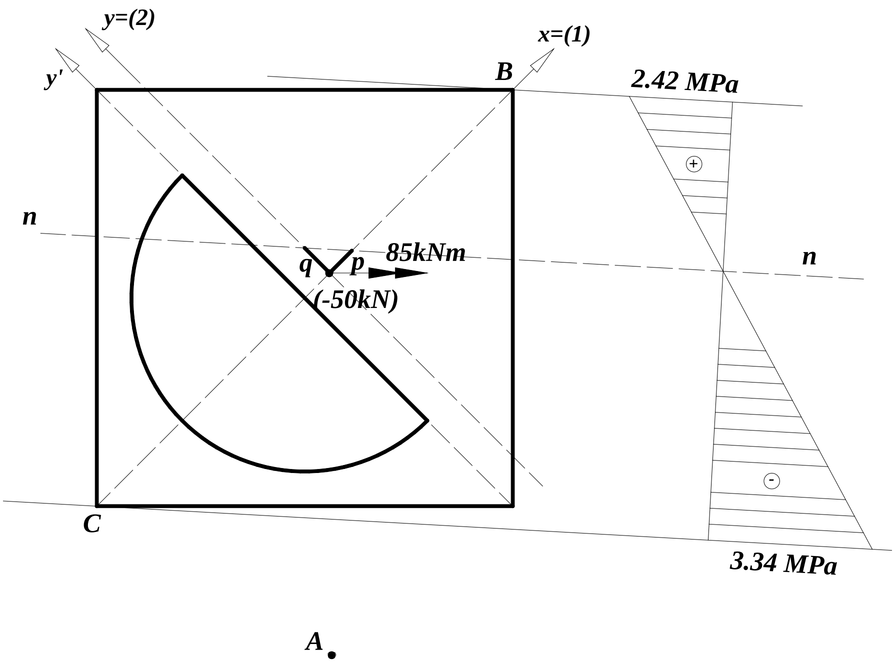
$$i_2^{\,2} = i_y^{\,2} = \frac{885\,168}{2618.252} = 338.1\,cm^2$$

Ekvivalentno opterećenje:

$$A(a;\,b) = A\left(\frac{\Sigma M_y}{P};\,\frac{\Sigma M_x}{P}\right)$$

$$A = \left(\frac{8500\frac{\sqrt{2}}{2}}{-50};\,\frac{8500\frac{\sqrt{2}}{2}}{-50}\right)$$

$$A = (-120.21;\,-120.21)\ cm$$



Neutralna linija:

$$p = -\frac{i_y^{\,2}}{a} = -\frac{338.1}{-121.21} = 2.812\ cm$$

$$q = -\frac{i_x^{\,2}}{b} = -\frac{353.9}{-121.21} = 2.944\ cm$$

Tačke u kojima se javljaju ekstremne vrijednosti normalnih napona u sistemu **glavnih osa *xy*:**

$$B(x_B;\,y_B) = \left(\left(30\sqrt{2}-3.978\right);\,0\right) = (38.448;\,0)\,cm$$

$$C(x_C;\,y_C) = \left(-\left(30\sqrt{2}+3.978\right);\,0\right) = (-46.404;\,0)\,cm$$

Ekstremni normalni naponi:

$$\sigma_z = \pm\frac{P}{A}\left(1+\frac{b}{i_x^{\,2}}\cdot y + \frac{a}{i_y^{\,2}}\cdot x\right)$$

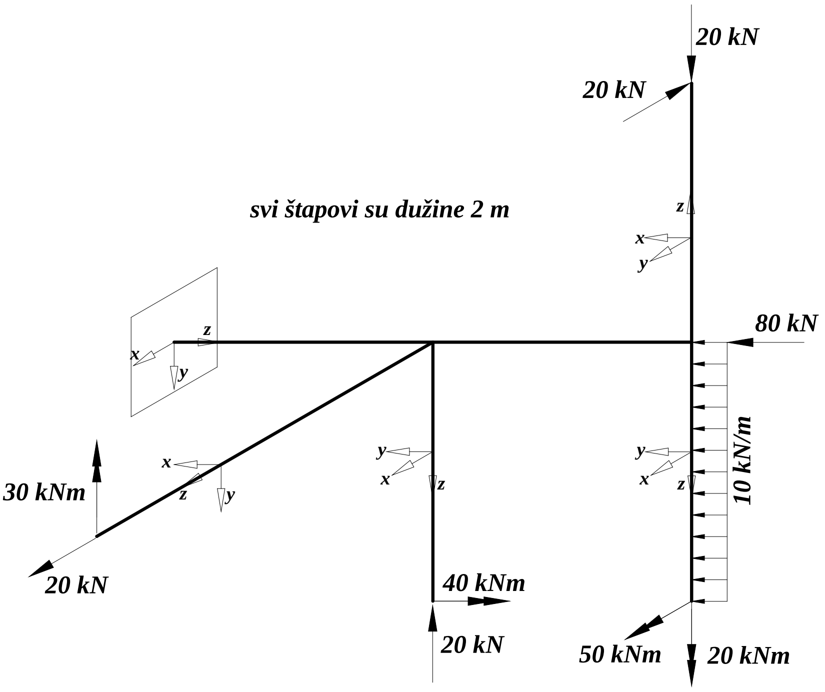
$$\sigma_z^{\,B} = -\frac{50}{2618.252}\left(1+\frac{-120.21}{353.9}\cdot 0 + \frac{-120.21}{338.1}\cdot 38.448\right) = 0.242\frac{kN}{cm^2}$$

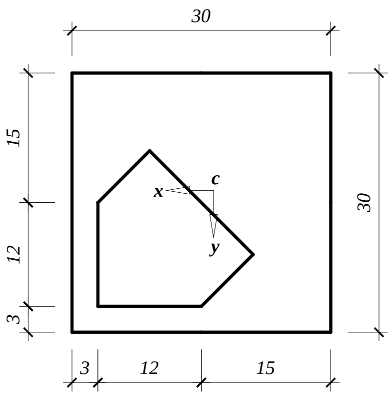
$$\sigma_z^{\,C} = -\frac{50}{2618.252}\left(1+\frac{-120.21}{353.9}\cdot 0 + \frac{-120.21}{338.1}\cdot(-46.404)\right) = -0.334\frac{kN}{cm^2}$$

3. Za nosač na slici:

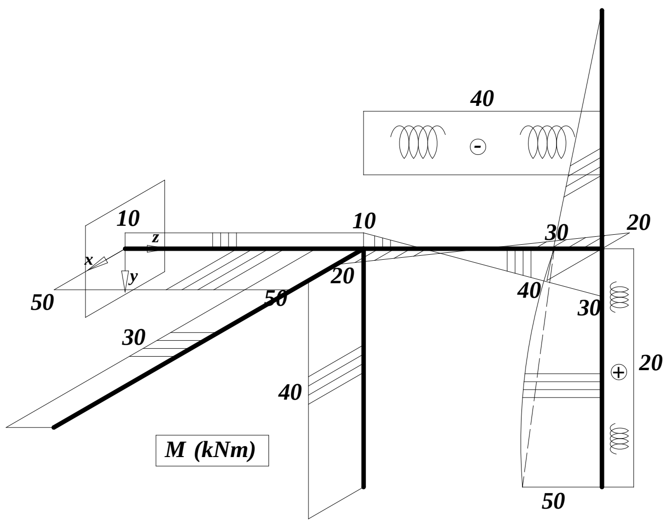
a. Nacrtati dijagrame presječnih sila (*M*, *T*, *N*);

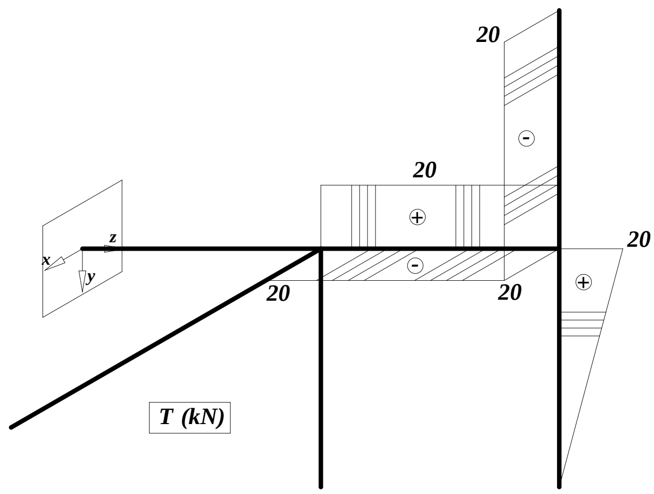
b. Za presjek u uklještenju nacrtati dijagram ekstremnih vrijednosti normalnog napona.

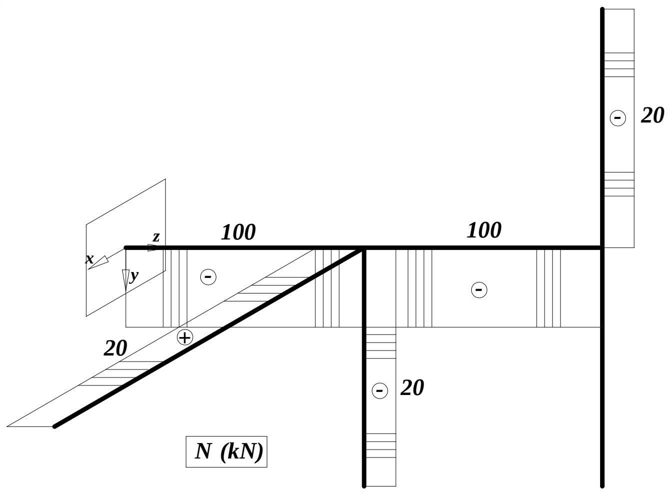




R J E Š E NJ E







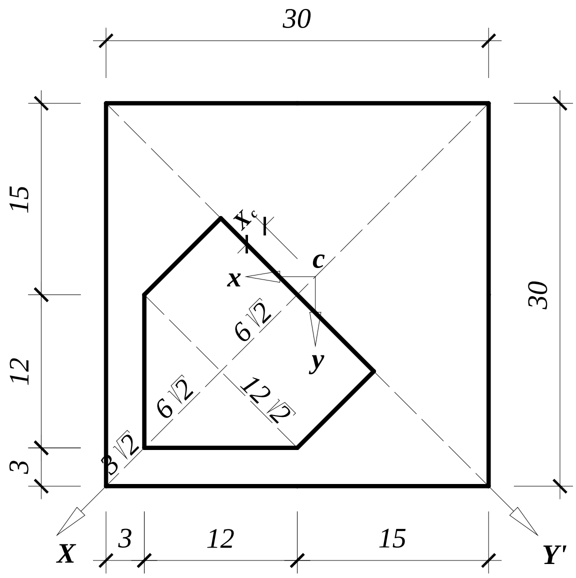
Geometrijske karakteristike poprečnog presjeka:

$C_1(0;0)\quad A_1 = 900\ cm^2$

$C_2(3\sqrt{2};0)\quad A_1 = 144\ cm^2$

$C_3(8\sqrt{2};0)\quad A_1 = 72\ cm^2$

$\Sigma A = 684\ cm^2$



$$X_C = \frac{-3\sqrt{2}\cdot 144 - 8\sqrt{2}\cdot 72}{684} = -2.084\ cm$$

$$I_X = \frac{1}{12}30^4 - \frac{1}{48}6\sqrt{2}\left(12\sqrt{2}\right)^3 - \frac{1}{12}6\sqrt{2}\left(12\sqrt{2}\right)^3 = 63\,180\ cm^4 \equiv I_1$$

$$I_Y = \frac{1}{12}30^4 + 30^2\cdot 2.084^2 - \left[\frac{1}{36}12\sqrt{2}\left(6\sqrt{2}\right)^3 + 72\cdot 13.40^2\right] - \left[\frac{1}{12}12\sqrt{2}\left(6\sqrt{2}\right)^3 + 144\cdot 6.327^2\right] = 51\,564\ cm^4 \equiv I_2$$

$$i_1^{\,2} = i_X^{\,2} = \frac{63\,180}{684} = 92.37\,cm^2$$

$$i_2^{\,2} = i_Y^{\,2} = \frac{51\,564}{684} = 75.386\,cm^2$$

Ekvivalentno opterećenje:

I način (određivanje koordinata napadne tačke ***A*** u sistemu osa ***xy***, a zatim rotacija koordinata da bi se dobile koordinate napadne tačke u sistemu **glavnih osa *12 (XY)***, tj ***a*** i ***b***)

$$A(x_A;\, y_A) = A\left(\frac{\Sigma M_y}{P};\frac{\Sigma M_x}{P}\right) = \left(\frac{50\cdot 10^2}{-100};\frac{-10\cdot 10^2}{-100}\right) = \left(-50;\,10\right)\ cm$$

$$X_A = x_A\cos\alpha + y_A\sin\alpha = (-50)\cos 45^0 + 10\sin 45^0 = -28.28\ cm \equiv a$$

$$Y_A = -x_A\sin\alpha + y_A\cos\alpha = 50\sin 45^0 + 10\cos 45^0 = 42.43\ cm \equiv b$$

$$\boldsymbol{A(X_A;\,Y_A) = \left(a;\,b\right) = \left(-28.28;\,42.43\right)\ cm}$$

II način (projektovanje vektora napadnih momenata na **glavne ose** i proračun koordinata napadne tačke ***A*** u sistemu **glavnih osa *12 (XY)***, tj ***a*** i ***b***)

$$\Sigma M_X = -\left(50\frac{\sqrt{2}}{2} + 10\frac{\sqrt{2}}{2}\right) = -42.43\ kNm$$

$$\Sigma M_Y = 50\frac{\sqrt{2}}{2} - 10\frac{\sqrt{2}}{2} = 28.28\ kNm$$

$$A(X_A;\,Y_A) = (a;\ b) = \left(\frac{\Sigma M_Y}{P};\frac{\Sigma M_X}{P}\right) = \left(\frac{28.28\cdot 10^2}{-100};\frac{-42.43\cdot 10^2}{-100}\right) = \left(-28.28;\,42.43\right)\ cm$$

Neutralna linija:

$$p = -\frac{i_Y^{\,2}}{a} = -\frac{75.386}{-28.28} = 2.666\ cm$$

$$q = -\frac{i_X^{\,2}}{b} = -\frac{92.37}{42.43} = -2.17\ cm$$

Tačke u kojima se javljaju ekstremne vrijednosti normalnih napona u sistemu glavnih osa ***XY:***

$$B(X_B; Y_B) = \left(2.084; 15\sqrt{2}\right)cm$$

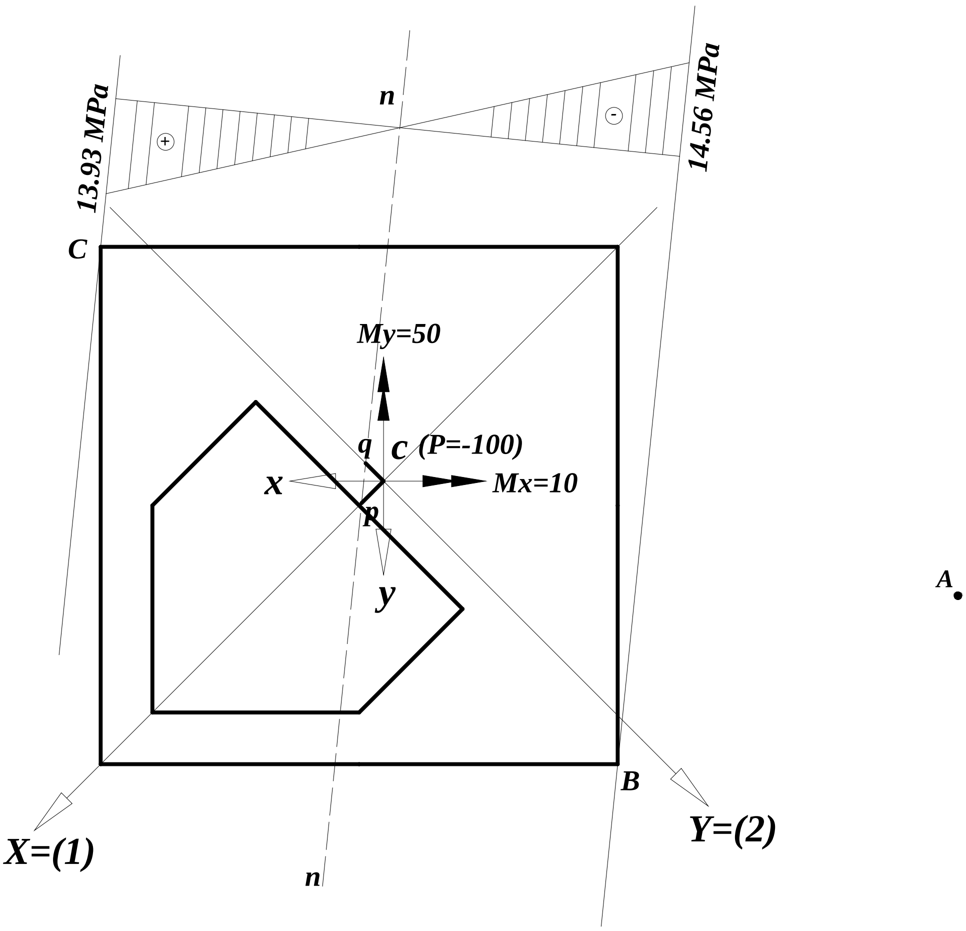
$$C(X_C; Y_C) = (2.084; -15\sqrt{2})cm$$

Ekstremni normalni naponi:

$$\sigma_z = \pm\frac{P}{A}\left(1 + \frac{b}{i_X^{\ 2}}\cdot Y + \frac{a}{i_Y^{\ 2}}\cdot X\right)$$

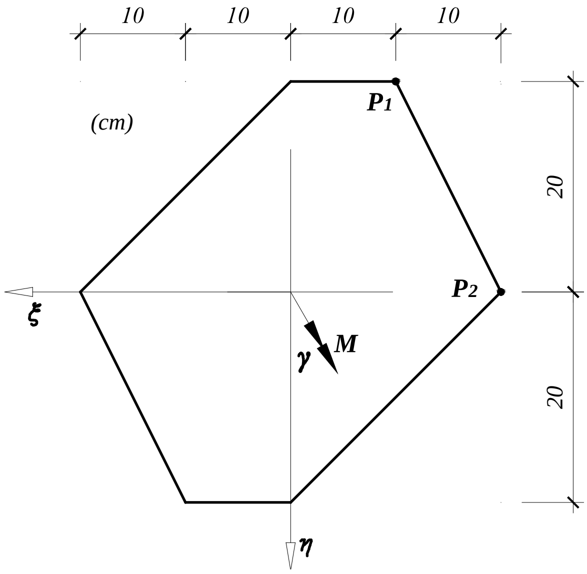
$$\sigma_z^{\ B} = -\frac{100}{684}\left(1 + \frac{42.43}{92.37}\cdot 15\sqrt{2} + \frac{-28.28}{75.386}\cdot 2.084\right) = -1.456\frac{kN}{cm^2}$$

$$\sigma_z^{\ C} = -\frac{100}{684}\left(1 + \frac{42.43}{92.37}\cdot\left(-15\sqrt{2}\right) + \frac{-28.28}{75.386}\cdot 2.084\right) = 1.393\frac{kN}{cm^2}$$



4. Betonski stub je opterećen kao na slici. Sračunati ekstremne normalne napone u poprečnom presjeku stuba i nacrtati djagram ekstremnih normalnih napona.

M=20 kNm
γ=30^0
P_1=-150 kN
P_2=+50 kN



R J E Š E NJ E

Geometrijske karakteristike poprečnog presjeka:

$$A = 40^2 - 2\frac{10\cdot 20}{2} - 2\frac{20\cdot 20}{2} = 1000\ cm^2$$

$$I_\xi = \frac{1}{12}40^4 - 2\left[\frac{1}{36}20^4 + 200\cdot 13.333^2\right] - 2\left[\frac{1}{36}10\cdot 20^3 + 100\cdot 13.333^2\right] = 93\ 333 cm^4$$

$$I_\eta = \frac{1}{12}40^4 - 2\left[\frac{1}{36}20^4 + 200\cdot 13.333^2\right] - 2\left[\frac{1}{36}10^3\cdot 20 + 100\cdot 16.667^2\right] = 76\ 667\ cm^4$$

$$I_{\xi\eta} = -2\left[\frac{20^4}{72} + 200\cdot 13.333\cdot(-13.333)\right] - 2\left[-\frac{10^2\cdot 20^2}{72} + 100\cdot 13.333\cdot 16.667\right] = 23\ 333\ cm^4$$

$$I_{1/2} = ... = 85\ 000 \pm 24\ 777$$

$$I_1 = 109\ 777\ cm^4 \equiv I_x$$

$$I_2 = 60\ 223 cm^4 \equiv I_y$$

$$tg2\alpha = \frac{-2\cdot 23\ 333}{76\ 667 - 23\ 333} = -2.8 \quad \rightarrow \quad \alpha = -35.173^0$$

$$i_1^{\ 2} = i_x^{\ 2} = \frac{109\ 777}{1000} = 109.777 cm^2$$

$$i_2^{\ 2} = i_y^{\ 2} = \frac{60\ 223}{1000} = 60.223\ cm^2$$

Ekvivalentno opterećenje:

$$\Sigma M_\xi = -20\sin 30^0 + 150\cdot 0.2 = 20\ kNm$$

$$\Sigma M_\eta = -20\cos 30^0 + 150\cdot 0.1 - 50\cdot 0.2 = -12.32\ kNm$$

$$\Sigma P = P_1 + P_2 = -150 + 50 = -100\ kN$$

$$A(\xi_A;\eta_A) = \left(\frac{\Sigma M_\eta}{\Sigma P};\frac{\Sigma M_\xi}{\Sigma P}\right) = \left(\frac{-12.32\cdot 10^2}{-100};\frac{20\cdot 10^2}{-100}\right) = (12.32;-20)\ cm$$

$$x_A = \xi_A\cos\alpha + \eta_A\sin\alpha = 12.32\cos\left(-35.173^0\right) - 20\sin\left(-35.173^0\right) = 21.59\ cm \equiv a$$

$$y_A = -\xi_A\sin\alpha + \eta_A\cos\alpha = -12.32\sin\left(-35.173^0\right) - 20\cos\left(-35.173^0\right) = -9.25\ cm \equiv b$$

$$\mathbf{A(x_A;\ y_A) = (a;\ b) = (21.59;\ -9.25)\ cm}$$

Neutralna linija:

$$p=-\frac{i_y^2}{a}=-\frac{60.223}{21.59}=-2.79\ cm$$

$$q=-\frac{i_x^2}{b}=-\frac{109.777}{-9.25}=11.88\ cm$$

Tačke u kojima se javljaju ekstremne vrijednosti normalnih napona u sistemu osa ***ξη*:**

$$B(\xi_B;\eta_B)=(0;-20)cm$$

$$C(\xi_C;\eta_C)=(0;20)cm$$

Tačke u kojima se javljaju ekstremne vrijednosti normalnih napona u sistemu glavnih osa ***xy*:**

$$x_B=\xi_B\cos\alpha+\eta_B\sin\alpha=0\cos\left(-35.173^0\right)-20\sin\left(-35.173^0\right)=11.521\ cm$$

$$y_B=-\xi_B\sin\alpha+\eta_B\cos\alpha=0\sin\left(-35.173^0\right)-20\cos\left(-35.173^0\right)=-16.348\ cm$$

$$x_C=\xi_C\cos\alpha+\eta_C\sin\alpha=0\cos\left(-35.173^0\right)+20\sin\left(-35.173^0\right)=-11.521\ cm$$

$$y_C=-\xi_C\sin\alpha+\eta_C\cos\alpha=0\sin\left(-35.173^0\right)+20\cos\left(-35.173^0\right)=16.348\ cm$$

$$\boldsymbol{B(x_B;\,y_B)=(11.521;-16.348)cm}$$

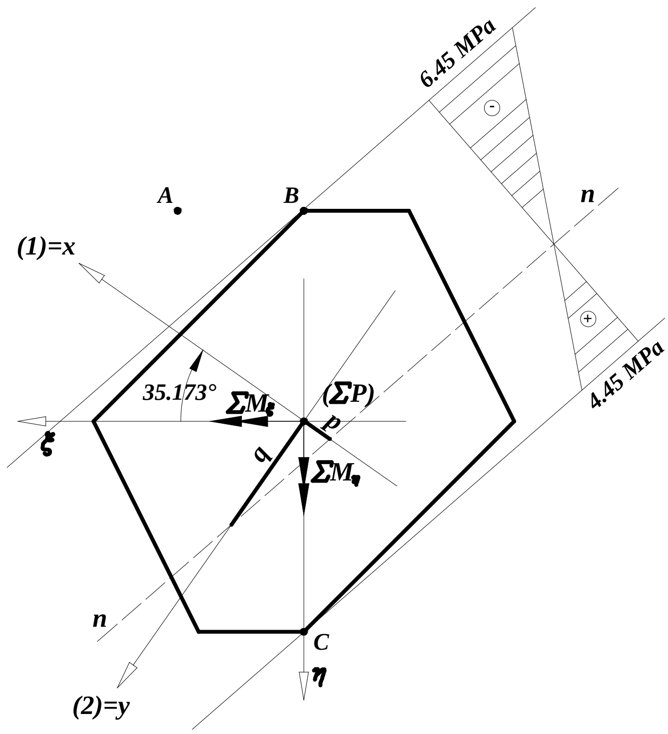
$$\boldsymbol{C(x_C;\,y_C)=(-11.521;16.348)cm}$$

Ekstremni normalni naponi:

$$\sigma_z=\pm\frac{\Sigma P}{A}\left(1+\frac{b}{i_x^2}\cdot y+\frac{a}{i_y^2}\cdot x\right)$$

$$\sigma_z^B=-\frac{100}{684}\left(1+\frac{-9.25}{109.777}\cdot(-16.348)+\frac{21.59}{60.223}\cdot 11.521\right)=-0.645\frac{kN}{cm^2}$$

$$\sigma_z^C=-\frac{100}{684}\left(1+\frac{-9.25}{109.777}\cdot 16.348+\frac{21.59}{60.223}\cdot(-11.521)\right)=0.445\frac{kN}{cm^2}$$



VJEŽBA BR. 14

TORZIJA

1. Za konzolni nosač na slici:

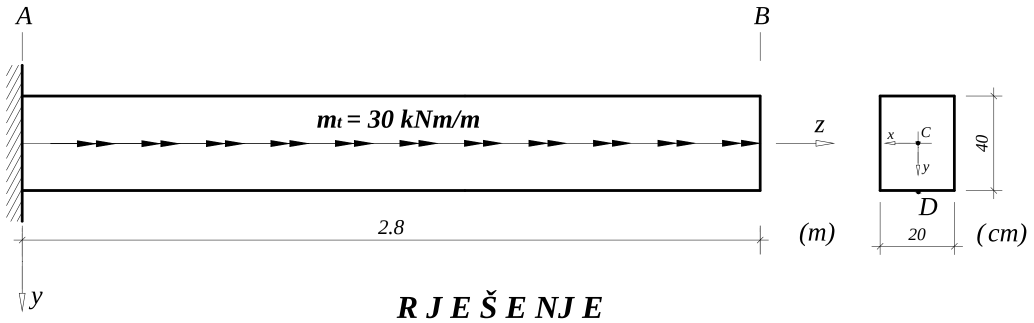
a. Nacrtati dijagram promjene momenta torzije duž nosača (M_t);

b. Sračunati maksimalni smičući napon (τ_{max}) i definisati presjek u kome on djeluje;

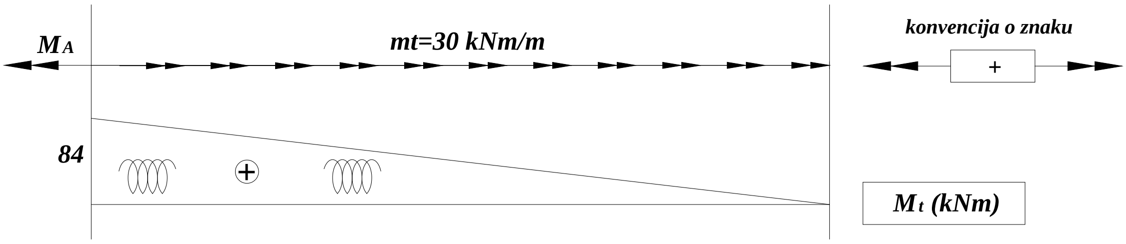
c. Nacrtati dijagram smičućih napona u presjeku u uklještenju;

d. Analizirati stanje napona u tački *D*;

e. Sračunati obrtanje poprečnog presjeka na slobodnom kraju nosača (φ_{BA}), ako je G=80 *GPa*.



R J E Š E NJ E

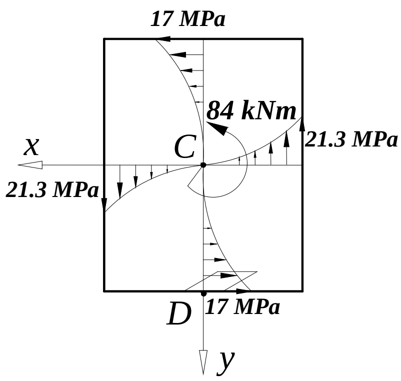


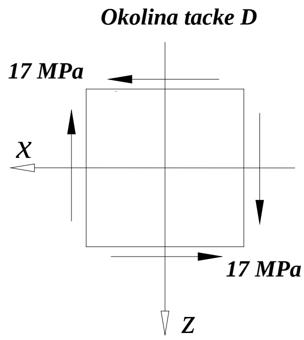
a. $M_A = m_t \cdot l = 30\,kNm/m \cdot 2.8\,m = 84\ kNm$

b. $\tau_{\max} = \dfrac{M_{t,\max}}{W_t};\ \ W_t = const\ \ (cont\ \ poprečni\ \ presjek) \ \rightarrow\ \tau_{\max}\ \ u\ \ presjeku\ u\ uklještenju$

$$\frac{h}{b} = \frac{40}{20} = 2 \ \rightarrow \begin{cases} \alpha = 0.457 \\ \beta = 0.493 \\ \gamma = 0.795 \end{cases} \ \rightarrow \ \begin{matrix} I_t = \alpha \cdot b^4 = 0.457 \cdot 20^4 = 7.312 \cdot 10^4\ cm^4 \\ W_t = \beta \cdot b^3 = 0.493 \cdot 20^3 = 3.944 \cdot 10^3\ cm^3 \end{matrix}$$

$$\tau_{\max} = \frac{M_{t,\max}}{W_t} = \frac{84 \cdot 10^2}{3.944 \cdot 10^3} = 2.13\ kN/cm^2$$





c. $\tau_{\max} = 2.13\ kN/cm^2$

$\tau' = \gamma \cdot \tau_{\max} = 0.795 \cdot 2.13 = 1.7\ kN/cm^2$

d. $\tau_{zx}^D = -1.7\ kN/cm^2$

e. $$\varphi_{BA} = \int_0^{2.8} \frac{84 - 30 \cdot z}{G \cdot I_t}\, dz = \frac{1}{G \cdot I_t}\left(84 \cdot z - 30\frac{z^2}{2}\right)\Bigg|_0^{2.8} =$$

$$= \frac{1}{G \cdot I_t}\left(84 \cdot 2.8 - 30\frac{2.8^2}{2}\right) = \frac{117.6\,kNm^2}{G \cdot I_t} = \frac{117.6 \cdot 10^4}{80 \cdot 10^2 \cdot 7.312 \cdot 10^4} = 2.01 \cdot 10^{-3}\ rad \cdot \frac{180}{\pi} = 0.115^0$$

2. Za konzolni nosač na slici:

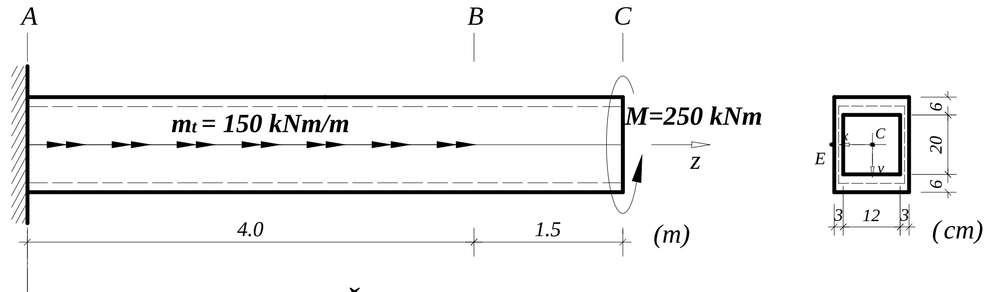
a. Nacrtati dijagram promjene momenta torzije duž nosača (M_t);

b. Sračunati maksimalni smičući napon (τ_{max}) i definisati presjek u kome on djeluje;

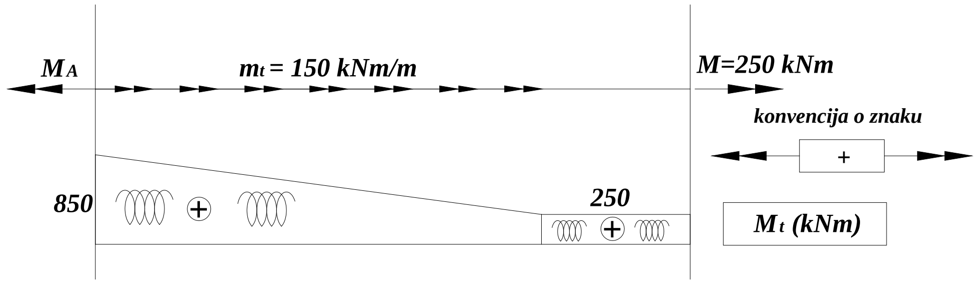
c. Nacrtati dijagram smičućih napona u presjeku na slobodnom kraju konzole;

d. Analizirati stanje napona u tački E;

e. Sračunati obrtanje poprečnog presjeka na slobodnom kraju konzole (φ_{CA}), ako je G=80 GPa.



R J E Š E NJ E



a. $M_A = m_t \cdot l + M = 150 \cdot 4 + 250 = 850\ kNm$

b. $\tau_{\max} = \dfrac{M_{t,\max}}{W_t}$ (τ_{max} djeluje u presjeku u kome djeluje maksimalni M_t, a to je presjek u uklještenju)

Geometrijske karakteristike poprečnog presjeka:

$$A_s = 26 \cdot 15 = 390\, cm^2$$

$$W_{t,i} = 2 \cdot A_s \cdot t_i$$

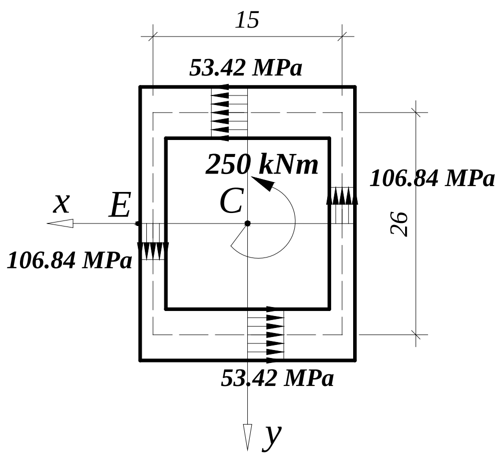
$$W_{t,1} = 2 \cdot 390 \cdot 6 = 4680\ cm^3$$

$$W_{t,2} = 2 \cdot 390 \cdot 3 = 2340\ cm^3$$

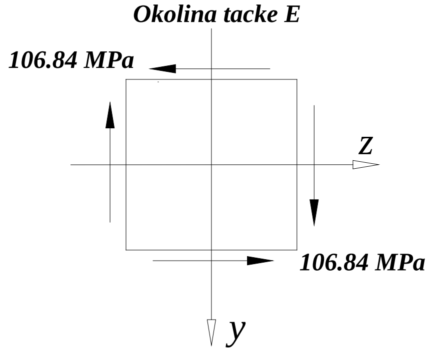
$$I_t = \frac{4 \cdot A_s^2}{\sum_i \frac{s_i}{t_i}} = \frac{4 \cdot 390^2}{\left(\frac{26}{3} + \frac{15}{6}\right) \cdot 2} = 27\,241\ cm^4$$

Maksimalni smičući napon:

$$\tau_{\max} = \frac{M_{t,\max}}{W_t^{\min}} = \frac{M_{t,\max}}{W_{t,2}} = \frac{850 \cdot 10^2}{2340} = 36.325\ kN/cm^2$$



Okolina tacke E



c. Presjek na slobodnom kraju konzole (presjek C):

$$M_t^C = 250\, kNm$$

$$\tau_C^{t=6} = \frac{M_t^C}{W_{t,1}} = \frac{250 \cdot 10^2}{4680} = 5.342\ kN/cm^2$$

$$\tau_C^{t=3} = \frac{M_t^C}{W_{t,2}} = \frac{250 \cdot 10^2}{2340} = 10.684\ kN/cm^2$$

d. $\tau_{zy}^E = +10.684\ kN/cm^2$

e. $\varphi_{CA} = \varphi_{BA} + \varphi_{CB}$

$$\varphi_{BA} = \int_0^4 \frac{850 - 150 \cdot z}{G \cdot I_t} dz = \frac{1}{G \cdot I_t}\left(850 \cdot z - 150\frac{z^2}{2}\right)\Bigg|_0^4 =$$

$$= \frac{1}{G \cdot I_t}\left(850 \cdot 4 - 150\frac{4^2}{2}\right) = \frac{2200 \cdot 10^4}{80 \cdot 10^2 \cdot 27\,241} = 0.1\, rad \cdot \frac{180}{\pi} = 5.73^0$$

$$\varphi_{CB} = \frac{M_{CB} \cdot l_{CB}}{G \cdot I_t} = \frac{250 \cdot 10^2 \cdot 1.5 \cdot 10^2}{80 \cdot 10^2 \cdot 27\,241} = 0.017\, rad \cdot \frac{180}{\pi} = 0.974^0$$

$$\varphi_{CA} = 0.1 + 0.017 = 0.117\, rad \cdot \frac{180}{\pi} = 6.7^0$$

3. Za obostrano uklijěsteni nosač na slici:

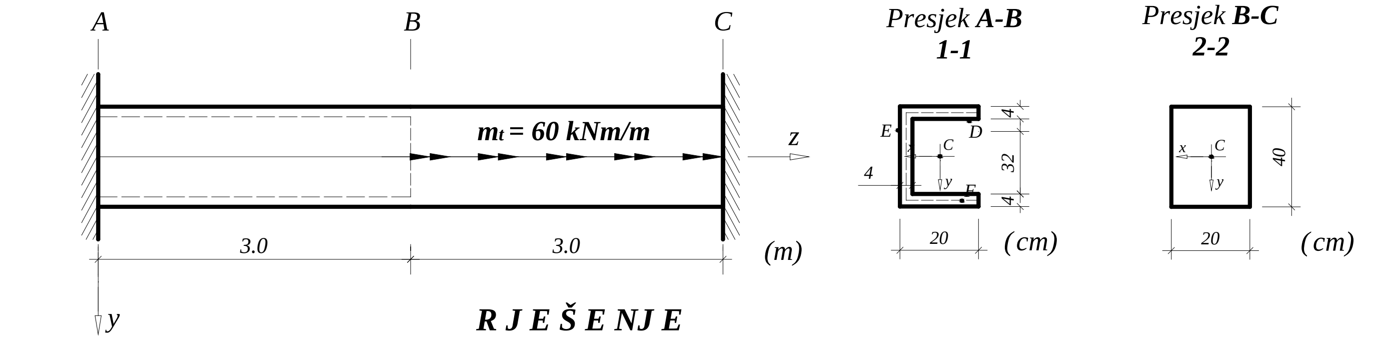
a. Nacrtati dijagram promjene momenta torzije duž nosača (M_t);

b. Sračunati maksimalni smičući napon (τ_{max}) i definisati presjek u kome on djeluje;

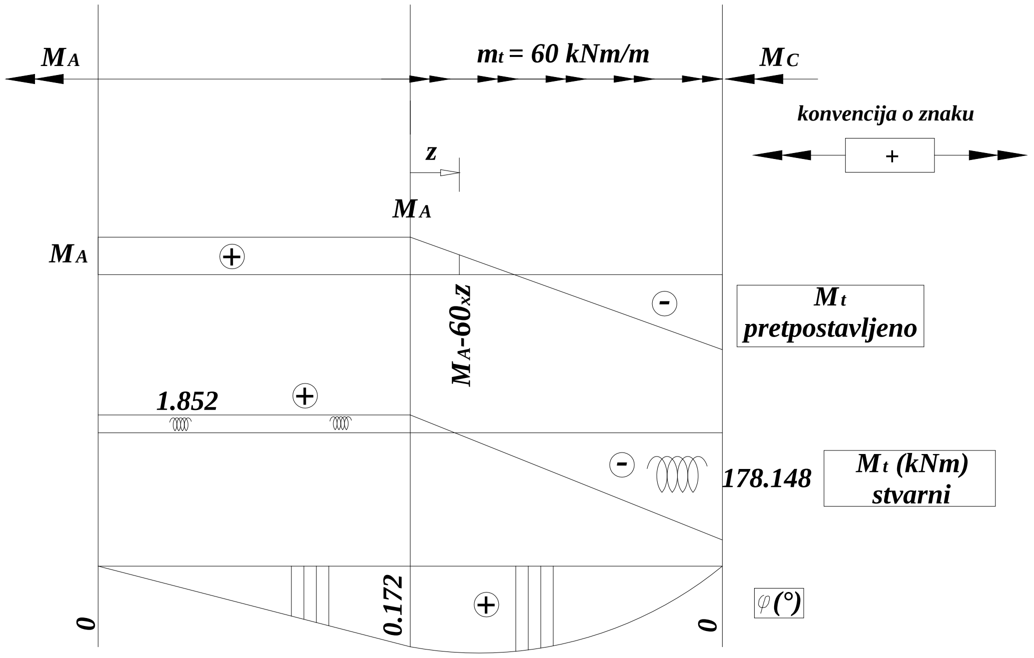
c. Nacrtati dijagram smičućih napona u presjeku *A*;

d. Analizirati stanje napona u tačkama *D, E* i *F*;

e. Nacrtati dijagram obratanja poprečnih presjeka duž nosača, ako su oba dijela (poprečna presjeka) izrađenana od istog materijala čiji je modul klizanja *G*=120 *GPa*.



R J E Š E NJ E



a. 2 statički nepoznate veličine (reaktivni torzioni momenti M_A i M_C)

1 raspoloživi uslov ravnoteže ($\sum M_t$=0)

Znači, 1x statički neodređen ptroblem.

1. Uslov ravnoteže $\sum M_t$=0 $\rightarrow$ $M_A + M_C = m_t \cdot l / 2$
2. Deformacijski uslov $\varphi_{CA} = 0$

Geometrijske karakteristike poprečnih presjeka:

Presjek 1-1:

$$I_t^{1-1} = \frac{1}{3}\sum_i h_i \cdot t_i^3 = \frac{1}{3}\left(18 \cdot 4^3 \cdot 2 + 36 \cdot 4^3\right) = 1536\ cm^4$$

$$W_{t,i} = \frac{I_t}{t_i} \quad (t_i = 4cm = const \rightarrow imamo\ sam\ 1\ otporni\ moment)$$

$$W_t^{1-1} = \frac{1536}{4} = 384\ cm^3$$

Presjek 2-2:

$$\frac{h}{b} = \frac{40}{20} = 2 \rightarrow \begin{cases} \alpha = 0.457 \\ \beta = 0.493 \\ \gamma = 0.795 \end{cases} \rightarrow \quad \begin{matrix} I_t^{2-2} = \alpha \cdot b^4 = 0.457 \cdot 20^4 = 7.312 \cdot 10^4\ cm^4 \\ W_t^{2-2} = \beta \cdot b^3 = 0.493 \cdot 20^3 = 3.944 \cdot 10^3\ cm^3 \end{matrix}$$

Proračun reakcija oslonaca:

$$\varphi_{CA} = \varphi_{BA} + \varphi_{CB} = 0$$

$$\frac{M_A \cdot l/2}{GI_t^{1-1}} + \int_0^{l/2} \frac{M_A - m_t \cdot z}{GI_t^{2-2}} dz = 0 \quad \Big| \cdot GI_t^{2-2}$$

$$\frac{I_t^{2-2}}{I_t^{1-1}} M_A \cdot l/2 + \left(M_A \cdot z - m_t \frac{z^2}{2}\right)\Bigg|_0^{l/2} = 0$$

$$\frac{I_t^{2-2}}{I_t^{1-1}} M_A \cdot l/2 + M_A \cdot l/2 - m_t \frac{(l/2)^2}{2} = 0 \quad \Big| /l/2$$

$$M_A\left(\frac{I_t^{2-2}}{I_t^{1-1}} + 1\right) = m_t \frac{l}{4}$$

$$M_A = \frac{m_t \cdot l}{4\left(\frac{I_t^{2-2}}{I_t^{1-1}} + 1\right)} = \frac{60 \cdot 6}{4\left(\frac{73120}{1536} + 1\right)} = 1.852\ kNm$$

Iz uslova ravnoteže $\rightarrow$ $M_C = m_t \cdot l/2 - M_A = 60 \cdot 3 - 1.852 = 178.148\ kNm$

Vidimo da je presjek 2-2 „navukao“ odnosno prihvatio skoro kompletno spoljašnje torziono opterećenje iz razloga što ima višestruko veću torzionu krutost koja se ogleda kroz geometrijsku komponentu krutosti odnosno torzioni moment inercije ($I_t^{2-2} = 73120 cm^4\ \ vs\ \ I_t^{1-1} = 1536 cm^4$).

b. Maksimalni smičući naponi:

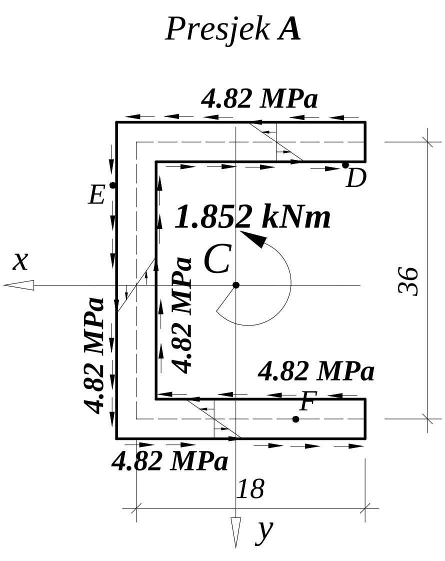
Presjek 1-1, dio *AB* ($M_{t,max}^{AB}$ =1.852 *kNm*):

$$\tau_{max}^{AB} = \frac{M_{t,max}^{AB}}{W_t^{1-1}} = \frac{1.852 \cdot 10^2}{384} = 0.482\ kN/cm^2$$

Presjek 2-2, dio *BC* ($M_{t,max}^{C}$ =178.148 *kNm*):

$$\tau_{max}^{BC} = \frac{M_{t,max}^{C}}{W_t^{2-2}} = \frac{178.148 \cdot 10^2}{3944} = 4.517\ kN/cm^2$$

$$\tau^C \equiv \tau_{max} \quad (djeluje\ u\ presjeku\ C)$$



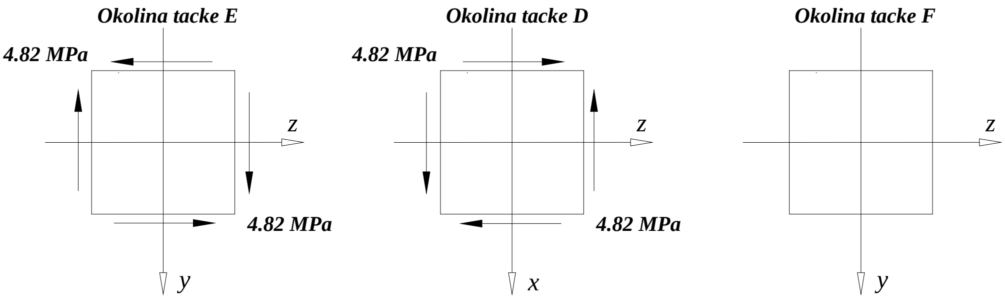
c. Smičući naponi u presjeku *A*:

$$M_t^A = 1.852 kNm \qquad \tau_A = 0.482\ kN/cm^2$$

d. $\tau_{zx}^{D} = -0.482\ kN/cm^2$

$\tau_{zy}^{E} = +0.482\ kN/cm^2$

$\tau_{zx}^{F} = 0$



e. $\varphi_{BA} = \dfrac{1.852 \cdot 10^2 \cdot 3 \cdot 10^2}{120 \cdot 10^2 \cdot 1536} = 0.003\ rad \cdot \dfrac{180}{\pi} = 0.172^0$

Kontrola

$$\varphi_{CB} = \int_0^3 \frac{1.852 - 60 \cdot z}{G \cdot I_t^{2-2}}\, dz = \frac{1}{G \cdot I_t^{2-2}} \left(1.852 \cdot z - 60 \frac{z^2}{2} \right) \Bigg|_0^3 =$$

$$= \frac{1}{G \cdot I_t^{2-2}} \left(1.852 \cdot 3 - 60 \frac{3^2}{2} \right) = \frac{-264.444 \cdot 10^4}{120 \cdot 10^2 \cdot 73120} = -0.003\ rad \cdot \frac{180}{\pi} = -0.172^0$$

$\varphi_{CA} = 0.003 - 0.003 = 0$

4. Na slici je prikazan obostrano ukliješteni nosač koji je izrađen od čelika (djelovi *AB* i *DE* - $G_č$=80 *GPa*) odnosno od bakra (dio *BD* – G_b=50 *GPa*).

a. Nacrtati dijagram promjene momenta torzije duž nosača (M_t);

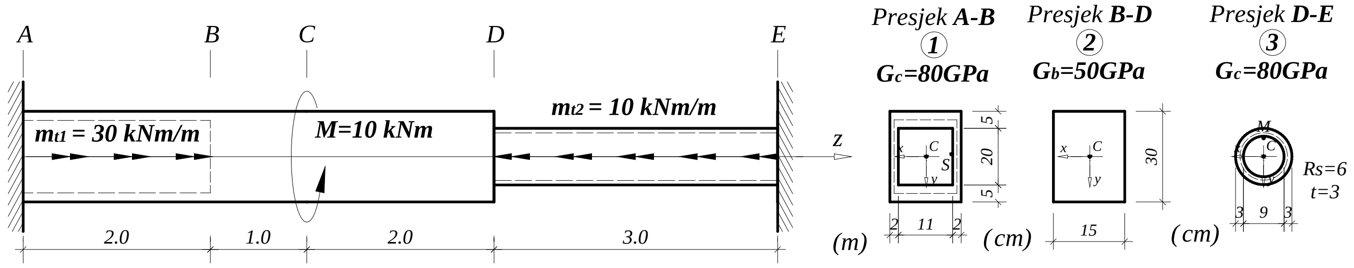
b. Nacrtati dijagram promjene maksimalnog napona (τ_{max}) duž nosača;

c. Nacrtati dijagram smičućih napona u presjecima *A* i D_{desno};

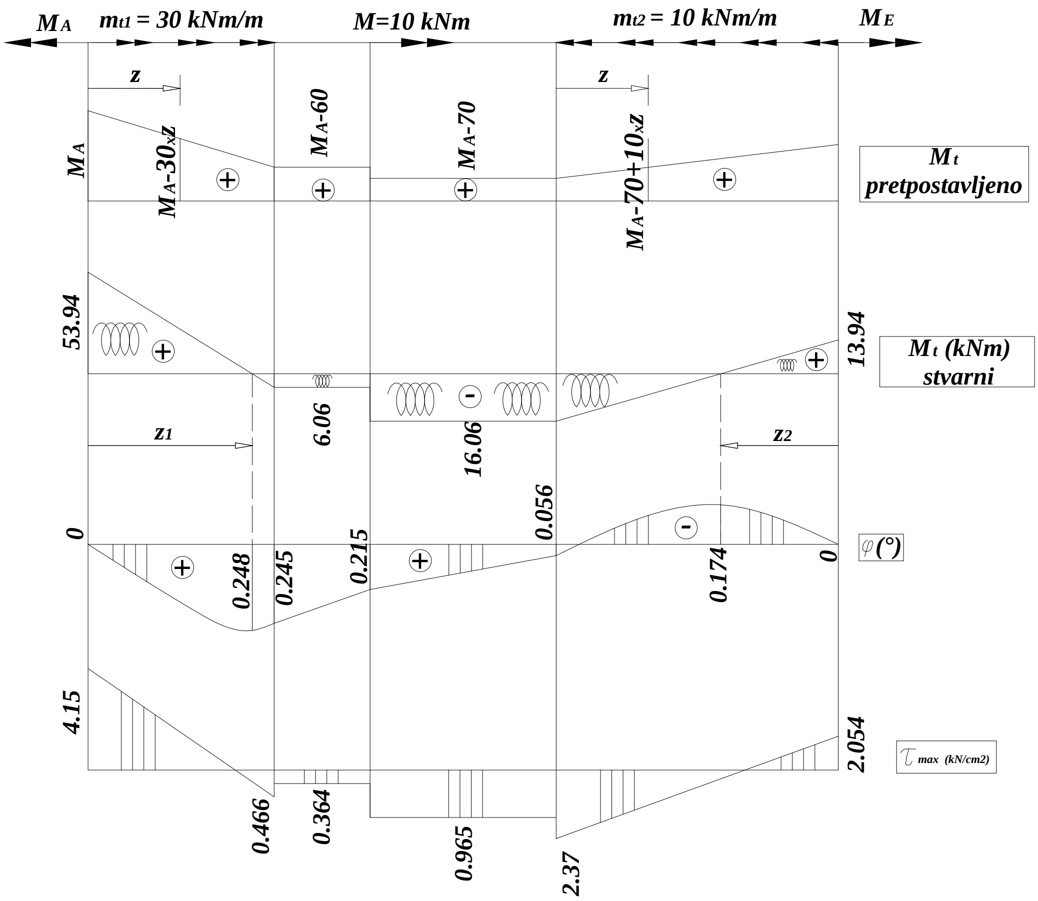
d. Analizirati stanje napona u tačkama *S* i *M*;

e. Nacrtati dijagram obratanja poprečnih presjeka duž nosača. Naznačiti obrtanja u presjecima *A, B, C, D* i *E*;

f. Sračunati maksimalno obrtanje po apsolutnoj vrijednosti i definisati presjek u kome se javlja.



R J E Š E NJ E



a. 2 statički nepoznate veličine (reaktivni torzioni momenti M_A i M_E)

1 raspoloživi uslov ravnoteže ($\sum M_t$=0)

Znači, 1x statički neodređen ptroblem.

1. Uslov ravnoteže $\sum M_t=0 \;\rightarrow\; M_C + m_t^{\boxed{1}} \cdot 2 + M_E = M_A + m_t^{\boxed{2}} \cdot 3$
2. Deformacijski uslov $\varphi_{EA} = 0$

Geometrijske karakteristike poprečnih presjeka:

Presjek 1:

$A_s = 13 \cdot 25 = 325 cm^2$

$W_{t,1}^{\boxed{1}} = 2 \cdot 325 \cdot 2 = 1300\ cm^3$

$W_{t,2}^{\boxed{1}} = 2 \cdot 325 \cdot 5 = 3250\ cm^3$

$$I_t^{\boxed{1}} = \frac{4 \cdot A_s^2}{\sum_i \frac{s_i}{t_i}} = \frac{4 \cdot 325^2}{\left(\frac{25}{2}+\frac{13}{5}\right)\cdot 2} = 13\,990\ cm^4$$

Presjek 2:

$$\frac{h}{b} = \frac{30}{15} = 2 \rightarrow \begin{cases} \alpha = 0.457 \\ \beta = 0.493 \\ \gamma = 0.795 \end{cases} \rightarrow \begin{matrix} I_t^{\boxed{2}} = \alpha \cdot b^4 = 0.457 \cdot 15^4 = 23136\ cm^4 \\ W_t^{\boxed{2}} = \beta \cdot b^3 = 0.493 \cdot 15^3 = 1664\ cm^3 \end{matrix}$$

Presjek 3:

$A_s = \pi \cdot 6^2 = 113.1 cm^2$

$W_t^{\boxed{3}} = 2 \cdot 113.1 \cdot 3 = 678.6\ cm^3$

$$I_t^{\boxed{1}} = \frac{4 \cdot A_s^2}{\sum_i \frac{s_i}{t_i}} = \frac{4 \cdot 113.1^2}{\frac{2\pi \cdot 6}{3}} = 4071.5\ cm^4$$

$$\frac{G_č}{G_b} = \frac{80}{50} = 1.6 \qquad \frac{I_t^{\boxed{1}}}{I_t^{\boxed{2}}} = \frac{13990}{23135.6} = 0.6047 \qquad \frac{I_t^{\boxed{1}}}{I_t^{\boxed{3}}} = \frac{13990}{4071.5} = 3.436$$

Proračun reakcija oslonaca:

$\varphi_{EA} = \varphi_{BA} + \varphi_{CB} + \varphi_{DC} + \varphi_{ED} = 0$

$$\int_0^2 \frac{M_A - 30 \cdot z}{G_č I_t^{\boxed{1}}} dz + \frac{(M_A - 60)\cdot 1}{G_b I_t^{\boxed{2}}} + \frac{(M_A - 70)\cdot 2}{G_b I_t^{\boxed{2}}} + \int_0^3 \frac{M_A - 70 + 10 \cdot z}{G_č I_t^{\boxed{3}}} dz = 0 \quad \Big| \cdot G_č I_t^{\boxed{1}}$$

$$\left(M_A \cdot z - 30\frac{z^2}{2} \right)\Bigg|_0^2 + \frac{G_č}{G_b}\frac{I_t^{\boxed{1}}}{I_t^{\boxed{2}}}(M_A - 60)\cdot 1 + \frac{G_č}{G_b}\frac{I_t^{\boxed{1}}}{I_t^{\boxed{2}}}(M_A - 70)\cdot 2 + \frac{I_t^{\boxed{1}}}{I_t^{\boxed{3}}}\left(M_A \cdot z - 70 \cdot z + 10\frac{z^2}{2} \right)\Bigg|_0^3 = 0$$

$$\left(M_A \cdot 2 - 30\frac{2^2}{2} \right) + 1.6 \cdot 0.6047(M_A - 60)\cdot 1 + 1.6 \cdot 0.6047(M_A - 70)\cdot 2 + 3.436\left(M_A \cdot 3 - 70 \cdot 3 + 10\frac{3^2}{2} \right) = 0$$

$M_A = 53.94\ kNm$

Iz uslova ravnoteže $\rightarrow$ $M_E = M_A + m_t^{\boxed{2}} \cdot 3 - M_C - m_t^{\boxed{1}} \cdot 2 = 53.94 + 10 \cdot 3 - 10 - 30 \cdot 2 = 13.94\ kNm$

b.

Maksimalni smičući naponi:

Presjek A $\left(M_t^A = 53.94 kNm\right)$: $\tau_{max}^A = \frac{M_t^A}{W_{t,1}^{\boxed{1}}} = \frac{53.94 \cdot 10^2}{1300} = 4.15\ kN/cm^2$

Presjek B_{lijevo} $\left(M_t^B = -6.06 kNm\right)$: $\tau_{max}^{B_{lijevo}} = \frac{M_t^B}{W_{t,1}^{\boxed{1}}} = \frac{-6.06 \cdot 10^2}{1300} = -0.466\ kN/cm^2$

Presjeci B_{desno} i C_{lijevo} $\left(M_t^{B_d, C_l} = -6.06 kNm\right)$: $\tau_{max}^{B_d, C_l} = \frac{M_t^{B_d, C_l}}{W_t^{\boxed{2}}} = \frac{-6.06 \cdot 10^2}{1664} = -0.364\ kN/cm^2$

Presjeci C_{desno} i D_{lijevo} $\left(M_t^{C_d,D_l} = -16.06\,kNm\right)$: $\tau_{\max}^{C_d,D_l} = \dfrac{M_t^{C_d,D_l}}{W_t^{\boxed{2}}} = \dfrac{-16.06 \cdot 10^2}{1664} = -0.965\ kN/cm^2$

Presjek D_{desno} $\left(M_t^{D_{desno}} = -16.06\,kNm\right)$: $\tau_{\max}^{D_{desno}} = \dfrac{M_t^{D_{desno}}}{W_t^{\boxed{3}}} = \dfrac{-16.06 \cdot 10^2}{678.6} = -2.37\ kN/cm^2$

Presjek E $\left(M_t^{E} = 13.94\,kNm\right)$: $\tau_{\max}^{E} = \dfrac{M_t^{E}}{W_t^{\boxed{3}}} = \dfrac{13.94 \cdot 10^2}{678.6} = 2.054\ kN/cm^2$

c. Smičući naponi u presjecima A i D_{desno}:

$$M_t^A = 53.94\,kNm$$

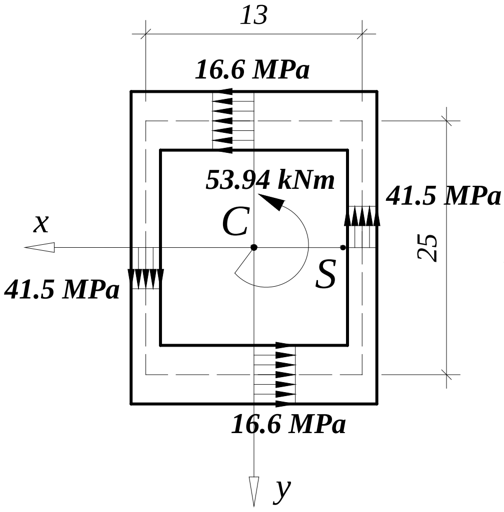
$$\tau_A^{t=2} = \frac{M_t^A}{W_{t,1}^{\boxed{1}}} = \frac{53.94 \cdot 10^2}{1300} = 4.15\ kN/cm^2$$

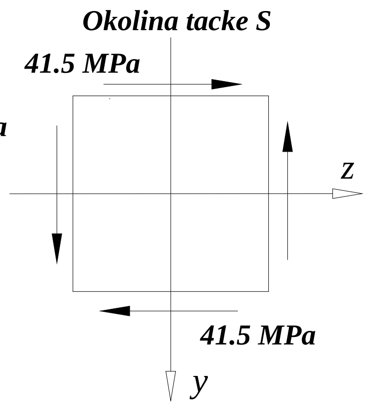
$$\tau_A^{t=5} = \frac{M_t^A}{W_{t,2}^{\boxed{1}}} = \frac{53.94 \cdot 10^2}{3250} = 1.66\ kN/cm^2$$

$$M_t^{D_{desno}} = -16.06\,kNm$$

$$\tau_{\max}^{D_{desno}} = \frac{M_t^{D_{desno}}}{W_t^{\boxed{3}}} = \frac{-16.06 \cdot 10^2}{678.6} = -2.37\ kN/cm^2$$

Presjek ***A***



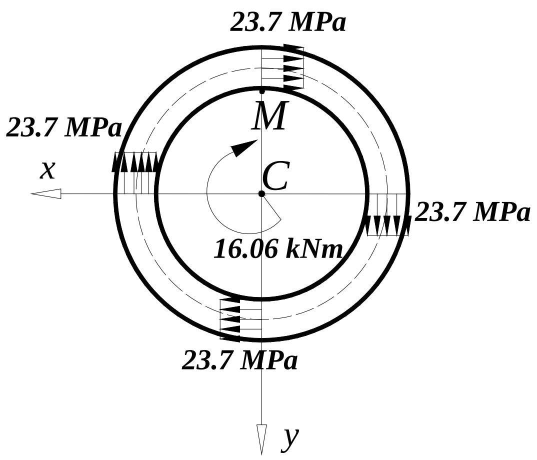


d.

$$\tau_{zy}^S = -4.15\ kN/cm^2$$

$$\tau_{zx}^M = -2.37\ kN/cm^2$$

Presjek ***D****desno*



e.

$$\varphi_{BA} = \int_0^2 \frac{53.94 - 30 \cdot z}{G_{\check{c}} \cdot I_t^{\boxed{1}}}\,dz = \frac{1}{G_{\check{c}} \cdot I_t^{\boxed{1}}}\left(53.94 \cdot z - 30\frac{z^2}{2}\right)\Bigg|_0^2 =$$

$$= \frac{1}{G_{\check{c}} \cdot I_t^{\boxed{1}}}\left(53.94 \cdot 2 - 30\frac{2^2}{2}\right) = \frac{47.88 \cdot 10^4}{80 \cdot 10^2 \cdot 13990} =$$

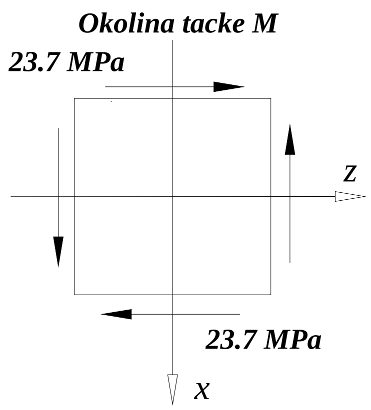
$$= 0.004278\ rad \cdot \frac{180}{\pi} = 0.245^0$$

$$\varphi_{CB} = \frac{M_{CB} \cdot l_{CB}}{G_b I_t^{\boxed{2}}} = \frac{-6.06 \cdot 10^2 \cdot 1 \cdot 10^2}{50 \cdot 10^2 \cdot 23135.6} = -0.0005239\,rad$$

$$\varphi_{CA} = \varphi_{BA} + \varphi_{CB} = 0.004278 - 0.0005239 =$$

$$= 0.003754\,rad \cdot \frac{180}{\pi} = 0.215^0$$

$$\varphi_{DC} = \frac{M_{DC} \cdot l_{DC}}{G_b I_t^{\boxed{2}}} = \frac{-16.06 \cdot 10^2 \cdot 2 \cdot 10^2}{50 \cdot 10^2 \cdot 23135.6} = -0.002777\,rad$$



$$\varphi_{DA} = \varphi_{CA} + \varphi_{DC} = 0.003754 - 0.002777 =$$

$$= 0.000977\, rad \cdot \frac{180}{\pi} = 0.056^0$$

$$\varphi_{ED} = \int_0^3 \frac{-16.06 + 10 \cdot z}{G_{\check{c}} \cdot I_t^{\boxed{3}}} dz = \frac{1}{G_{\check{c}} \cdot I_t^{\boxed{3}}} \left(-16.06 \cdot z + 10 \frac{z^2}{2} \right) \Bigg|_0^3 =$$

$$= \frac{1}{G_{\check{c}} \cdot I_t^{\boxed{3}}} \left(-16.06 \cdot 3 + 10 \frac{3^2}{2} \right) = \frac{-3.18 \cdot 10^4}{80 \cdot 10^2 \cdot 4071.5} = -0.000977\, rad$$

Kontrola:

$$\varphi_{EA} = \varphi_{DA} + \varphi_{ED} = 0.000977 - 0.000977 = 0$$

f. $\varphi(z) = \int_0^l \frac{M(z)}{GI_t} dz$

$$\max \varphi = \varphi(z_{\varphi_{\max}}) = ? \quad \frac{d\varphi(z)}{dz} = 0 \quad \frac{d\varphi(z)}{dz} = \frac{d}{dz} \left(\int_0^l \frac{M(z)}{GI_t} dz \right) = \frac{M(z)}{GI_t} = 0 \rightarrow M(z) = 0 \rightarrow \boxed{z_{\varphi_{\max}}}$$

Sa dijagrama momenata torzije (M_t) se mogu uočiti dva presjeka u kojima je moment torzije jednak 0.

Položaj predmetnih presjeka ćemo odrediti iz sličnosti trouglova:

$$\frac{53.94 + 6.06}{2} = \frac{53.94}{z_1} \rightarrow z_1 = 1.798m$$

$$\frac{13.94 + 16.06}{3} = \frac{13.94}{z_2} \rightarrow z_2 = 1.394m$$

Obrtanje presjeka definisanih sa z_1 i z_2:

$$\varphi_{\max} = \varphi(z_1) = \int_0^{1.798} \frac{53.94 - 30 \cdot z}{G_{\check{c}} \cdot I_t^{\boxed{1}}} dz = \frac{1}{G_{\check{c}} \cdot I_t^{\boxed{1}}} \left(53.94 \cdot z - 30 \frac{z^2}{2} \right) \Bigg|_0^{1.798} =$$

$$= \frac{1}{G_{\check{c}} \cdot I_t^{\boxed{1}}} \left(53.94 \cdot 1.798 - 30 \frac{1.798^2}{2} \right) = \frac{48.492 \cdot 10^4}{80 \cdot 10^2 \cdot 13990} = 0.00433\, rad \cdot \frac{180}{\pi} = 0.248^0 \equiv \varphi_{\max} - apsolutni \max$$

$$\varphi_{\max} = \varphi(z_2) = \int_0^{1.394} \frac{13.94 - 10 \cdot z}{G_{\check{c}} \cdot I_t^{\boxed{3}}} dz = \frac{1}{G_{\check{c}} \cdot I_t^{\boxed{3}}} \left(13.94 \cdot z - 10 \frac{z^2}{2} \right) \Bigg|_0^{1.394} =$$

$$= \frac{1}{G_{\check{c}} \cdot I_t^{\boxed{3}}} \left(13.94 \cdot 1.394 - 10 \frac{1.394^2}{2} \right) = \frac{9.716 \cdot 10^4}{80 \cdot 10^2 \cdot 4071.5} = 0.00298\, rad \cdot \frac{180}{\pi} = 0.171^0 \equiv \varphi_{\max} - lokalni \max$$